23
Appendices

A. Physical Constants

We use SI units throughout this text. Simple ways to convert between SI and other popular units, such as Gaussian, may be found in Refs. [100–103].

The Committee on Data for Science and Technology (CODATA) of NIST maintains the values of many physical constants [89]. The most current values can be obtained from the CODATA web site [1299]. Some commonly used constants are listed below:

quantity	symbol	value	units
speed of light in vacuum	c_0, c	$299\,792\,458$	$\mathrm{m\,s^{-1}}$
permittivity of vacuum	ϵ_0	$8.854\,187\,817\times 10^{-12}$	$\mathrm{F\ m^{-1}}$
permeability of vacuum	μ_0	$4\pi\times 10^{-7}$	$\mathrm{H\ m^{-1}}$
characteristic impedance	η_0, Z_0	$376.730\,313\,461$	Ω
electron charge	e	$1.602\,176\,462\times 10^{-19}$	C
electron mass	m_e	$9.109\,381\,887\times 10^{-31}$	kg
Boltzmann constant	k	$1.380\,650\,324\times 10^{-23}$	$\mathrm{J\,K^{-1}}$
Avogadro constant	N_A, L	$6.022\,141\,994\times 10^{23}$	$\mathrm{mol^{-1}}$
Planck constant	h	$6.626\,068\,76\times 10^{-34}$	J/Hz
Gravitational constant	G	$6.672\,59\times 10^{-11}$	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Earth mass	$M_\oplus$	5.972×10^{24}	kg
Earth equatorial radius	a_e	6378	km

In the table, the constants c, μ_0 are taken to be exact, whereas ϵ_0, η_0 are derived from the relationships:

$$\epsilon_0 = \frac{1}{\mu_0 c^2}\,, \quad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \mu_0 c$$

The energy unit of electron volt (eV) is defined to be the work done by an electron in moving across a voltage of one volt, that is, $1\ \mathrm{eV} = 1.602\,176\,462\times 10^{-19}\ \mathrm{C}\cdot 1\ \mathrm{V}$, or

$$1\ \mathrm{eV} = 1.602\,176\,462\times 10^{-19}\ \mathrm{J}$$

In units of eV/Hz, Planck's constant h is:

$$h = 4.135\,667\,27 \times 10^{-15}\ \text{eV/Hz} = 1\ \text{eV}/241.8\ \text{THz}$$

that is, 1 eV corresponds to a frequency of 241.8 THz, or a wavelength of 1.24 μm.

B. Electromagnetic Frequency Bands

The ITU† divides the radio frequency (RF) spectrum into the following frequency and wavelength bands in the range from 30 Hz to 3000 GHz:

RF Spectrum					
	band designations	**frequency**		**wavelength**	
ELF	Extremely Low Frequency	30–300	Hz	1–10	Mm
VF	Voice Frequency	300–3000	Hz	100–1000	km
VLF	Very Low Frequency	3–30	kHz	10–100	km
LF	Low Frequency	30–300	kHz	1–10	km
MF	Medium Frequency	300–3000	kHz	100–1000	m
HF	High Frequency	3–30	MHz	10–100	m
VHF	Very High Frequency	30–300	MHz	1–10	m
UHF	Ultra High Frequency	300–3000	MHz	10–100	cm
SHF	Super High Frequency	3–30	GHz	1–10	cm
EHF	Extremely High Frequency	30–300	GHz	1–10	mm
	Submillimeter	300-3000	GHz	100–1000	μm

An alternative subdivision of the low-frequency bands is to designate the bands 3–30 Hz, 30–300 Hz, and 300–3000 Hz as extremely low frequency (ELF), super low frequency (SLF), and ultra low frequency (ULF), respectively.

Microwaves span the 300 MHz–300 GHz frequency range. Typical microwave and satellite communication systems and radar use the 1–30 GHz band. The 30–300 GHz EHF band is also referred to as the millimeter band.

The 1–100 GHz range is subdivided further into the subbands shown on the right.

Microwave Bands		
band	**frequency**	
L	1–2	GHz
S	2–4	GHz
C	4–8	GHz
X	8–12	GHz
Ku	12–18	GHz
K	18–27	GHz
Ka	27–40	GHz
V	40–75	GHz
W	80–100	GHz

Some typical RF applications are as follows. AM radio is broadcast at 535–1700 kHz falling within the MF band. The HF band is used in short-wave radio, navigation, amateur, and CB bands. FM radio at 88–108 MHz, ordinary TV, police, walkie-talkies, and remote control occupy the VHF band.

Cell phones, personal communication systems (PCS), pagers, cordless phones, global positioning systems (GPS), RF identification systems (RFID), UHF-TV channels, microwave ovens, and long-range surveillance radar fall within the UHF band.

†International Telecommunication Union.

The SHF microwave band is used in radar (traffic control, surveillance, tracking, missile guidance, mapping, weather), satellite communications, direct-broadcast satellite (DBS), and microwave relay systems. Multipoint multichannel (MMDS) and local multipoint (LMDS) distribution services, fall within UHF and SHF at 2.5 GHz and 30 GHz.

Industrial, scientific, and medical (ISM) bands are within the UHF and low SHF, at 900 MHz, 2.4 GHz, and 5.8 GHz. Radio astronomy occupies several bands, from UHF to L–W microwave bands.

Beyond RF, come the infrared (IR), visible, ultraviolet (UV), X-ray, and γ-ray bands. The IR range extends over 3–300 THz, or 1–100 μm. Many IR applications fall in the 1–20 μm band. For example, optical fiber communications typically use laser light at 1.55 μm or 193 THz because of the low fiber losses at that frequency. The UV range lies beyond the visible band, extending typically over 10–400 nm.

band	wavelength	frequency	energy
infrared	100–1 μm	3–300 THz	
ultraviolet	400–10 nm	750 THz–30 PHz	
X-Ray	10 nm–100 pm	30 PHz–3 EHz	0.124–124 keV
γ-ray	< 100 pm	> 3 EHz	> 124 keV

The CIE† defines the visible spectrum to be the wavelength range 380–780 nm, or 385–789 THz. Colors fall within the following typical wavelength/frequency ranges:

Visible Spectrum		
color	**wavelength**	**frequency**
red	780–620 nm	385–484 THz
orange	620–600 nm	484–500 THz
yellow	600–580 nm	500–517 THz
green	580–490 nm	517–612 THz
blue	490–450 nm	612–667 THz
violet	450–380 nm	667–789 THz

X-ray frequencies fall in the PHz (petahertz) range and γ-ray frequencies in the EHz (exahertz) range.‡ X-rays and γ-rays are best described in terms of their energy, which is related to frequency through Planck's relationship, $E = hf$. X-rays have typical energies of the order of keV, and γ-rays, of the order of MeV and beyond. By comparison, photons in the visible spectrum have energies of a couple of eV.

The earth's atmosphere is mostly opaque to electromagnetic radiation, except for three significant "windows", the visible, the infrared, and the radio windows. These three bands span the wavelength ranges of 380-780 nm, 1-12 μm, and 5 mm–20 m, respectively.

Within the 1-10 μm infrared band there are some narrow transparent windows. For the rest of the IR range (1–1000 μm), water and carbon dioxide molecules absorb infrared radiation—this is responsible for the Greenhouse effect. There are also some minor transparent windows for 17–40 and 330–370 μm.

†Commission Internationale de l'Eclairage (International Commission on Illumination.)

‡1 THz = 10^{12} Hz, 1 PHz = 10^{15} Hz, 1 EHz = 10^{18} Hz.

Beyond the visible band, ultraviolet and X-ray radiation are absorbed by ozone and molecular oxygen (except for the ozone holes.)

C. Vector Identities and Integral Theorems

Algebraic Identities

$$|\boldsymbol{A}|^2|\boldsymbol{B}|^2 = |\boldsymbol{A}\cdot\boldsymbol{B}|^2 + |\boldsymbol{A}\times\boldsymbol{B}|^2 \tag{C.1}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = (\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A} = (\boldsymbol{C}\times\boldsymbol{A})\cdot\boldsymbol{B} \tag{C.2}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\,(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}\,(\boldsymbol{A}\cdot\boldsymbol{B}) \qquad \text{(BAC-CAB rule)} \tag{C.3}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\cdot\boldsymbol{C})\,(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{A}\cdot\boldsymbol{D})\,(\boldsymbol{B}\cdot\boldsymbol{C}) \tag{C.4}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D}) = \big[(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{D}\big]\boldsymbol{C} - \big[(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}\big]\boldsymbol{D} \tag{C.5}$$

$$\boldsymbol{A} = \hat{\mathbf{n}}\times(\boldsymbol{A}\times\hat{\mathbf{n}}) + (\hat{\mathbf{n}}\cdot\boldsymbol{A})\hat{\mathbf{n}} = \boldsymbol{A}_\perp + \boldsymbol{A}_\parallel \tag{C.6}$$

where $\hat{\mathbf{n}}$ is any unit vector, and $\boldsymbol{A}_\perp$, $\boldsymbol{A}_\parallel$ are the components of $\boldsymbol{A}$ perpendicular and parallel to $\hat{\mathbf{n}}$. Note also that $\hat{\mathbf{n}}\times(\boldsymbol{A}\times\hat{\mathbf{n}}) = (\hat{\mathbf{n}}\times\boldsymbol{A})\times\hat{\mathbf{n}}$. A three-dimensional vector can equally well be represented as a column vector:

$$\boldsymbol{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} \quad\Leftrightarrow\quad \boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ b_z \end{bmatrix} \tag{C.7}$$

Consequently, the dot and cross products may be represented in matrix form:

$$\boldsymbol{a}\cdot\boldsymbol{b} \quad\Leftrightarrow\quad \boldsymbol{a}^T\boldsymbol{b} = [a_x, a_y, a_z]\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z \tag{C.8}$$

$$\boldsymbol{a}\times\boldsymbol{b} \quad\Leftrightarrow\quad A\boldsymbol{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \tag{C.9}$$

The cross-product matrix A satisfies the following identity:

$$A^2 = \boldsymbol{a}\boldsymbol{a}^T - (\boldsymbol{a}^T\boldsymbol{a})I \tag{C.10}$$

where I is the 3×3 identity matrix. Applied to a unit vector $\hat{\mathbf{n}}$, this identity reads:

$$I = \hat{\mathbf{n}}\hat{\mathbf{n}}^T - \hat{N}^2, \quad \text{where} \quad \hat{\mathbf{n}} = \begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}, \quad \hat{\mathbf{n}}^T\hat{\mathbf{n}} = 1 \tag{C.11}$$

This corresponds to the matrix form of the parallel/transverse decomposition (C.6). Indeed, we have $\boldsymbol{a}_\parallel = \hat{\mathbf{n}}(\hat{\mathbf{n}}^T\boldsymbol{a})$ and $\boldsymbol{a}_\perp = (\hat{\mathbf{n}}\times\boldsymbol{a})\times\hat{\mathbf{n}} = -\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{a}) = -\hat{N}(\hat{N}\boldsymbol{a}) = -\hat{N}^2\boldsymbol{a}$. Therefore, $\boldsymbol{a} = I\boldsymbol{a} = (\hat{\mathbf{n}}\hat{\mathbf{n}}^T - \hat{N}^2)\,\boldsymbol{a} = \boldsymbol{a}_\parallel + \boldsymbol{a}_\perp$.

Differential Identities

$$\nabla\times(\nabla\psi) = 0 \tag{C.12}$$

$$\nabla\cdot(\nabla\times\boldsymbol{A}) = 0 \tag{C.13}$$

$$\nabla\cdot(\psi\boldsymbol{A}) = \boldsymbol{A}\cdot\nabla\psi + \psi\nabla\cdot\boldsymbol{A} \tag{C.14}$$

$$\nabla\times(\psi\boldsymbol{A}) = \psi\nabla\times\boldsymbol{A} + \nabla\psi\times\boldsymbol{A} \tag{C.15}$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) \tag{C.16}$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \tag{C.17}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{C.18}$$

$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} \tag{C.19}$$

$$A_x\nabla B_x + A_y\nabla B_y + A_z\nabla B_z = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) \tag{C.20}$$

$$B_x\nabla A_x + B_y\nabla A_y + B_z\nabla A_z = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) \tag{C.21}$$

$$(\hat{\mathbf{n}}\times\nabla)\times\boldsymbol{A} = \hat{\mathbf{n}}\times(\nabla\times\boldsymbol{A}) + (\hat{\mathbf{n}}\cdot\nabla)\boldsymbol{A} - \hat{\mathbf{n}}(\nabla\cdot\boldsymbol{A}) \tag{C.22}$$

$$\begin{aligned} \psi(\hat{\mathbf{n}}\cdot\nabla)\boldsymbol{E} - \boldsymbol{E}(\hat{\mathbf{n}}\cdot\nabla\psi) = &\big[(\hat{\mathbf{n}}\cdot\nabla)(\psi\boldsymbol{E}) + \hat{\mathbf{n}}\times\big(\nabla\times(\psi\boldsymbol{E})\big) - \hat{\mathbf{n}}\nabla\cdot(\psi\boldsymbol{E})\big] \\ &+ \big[\hat{\mathbf{n}}\,\psi\,\nabla\cdot\boldsymbol{E} - (\hat{\mathbf{n}}\times\boldsymbol{E})\times\nabla\psi - \psi\,\hat{\mathbf{n}}\times(\nabla\times\boldsymbol{E}) - (\hat{\mathbf{n}}\cdot\boldsymbol{E})\,\nabla\psi\big] \end{aligned} \tag{C.23}$$

With $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, and the unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$, we have:

$$\nabla r = \hat{\mathbf{r}}, \quad \nabla r^2 = 2\mathbf{r}, \quad \nabla\frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2}, \quad \nabla\cdot\mathbf{r} = 3, \quad \nabla\times\mathbf{r} = 0, \quad \nabla\cdot\hat{\mathbf{r}} = \frac{2}{r} \tag{C.24}$$

Integral Theorems for Closed Surfaces

The theorems involve a volume V surrounded by a closed surface S. The divergence or Gauss' theorem is:

$$\boxed{\int_V \nabla\cdot\boldsymbol{A}\,dV = \oint_S \boldsymbol{A}\cdot\hat{\mathbf{n}}\,dS} \qquad \text{(Gauss' divergence theorem)} \tag{C.25}$$

where $\hat{\mathbf{n}}$ is the *outward* normal to the surface. Green's first and second identities are:

$$\int_V \big[\varphi\nabla^2\psi + \nabla\varphi\cdot\nabla\psi\big]\,dV = \oint_S \varphi\frac{\partial\psi}{\partial n}\,dS \tag{C.26}$$

$$\int_V \big[\varphi\nabla^2\psi - \psi\nabla^2\varphi\big]\,dV = \oint_S \Big(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\Big)\,dS \tag{C.27}$$

where $\dfrac{\partial}{\partial n} = \hat{\mathbf{n}}\cdot\nabla$ is the directional derivative along $\hat{\mathbf{n}}$. Some related theorems are:

$$\int_V \nabla^2\psi\, dV = \oint_S \hat{\mathbf{n}}\cdot\nabla\psi\, dS = \oint_S \frac{\partial\psi}{\partial n}\, dS \tag{C.28}$$

$$\int_V \nabla\psi\, dV = \oint_S \psi\,\hat{\mathbf{n}}\, dS \tag{C.29}$$

$$\int_V \nabla^2 \boldsymbol{A}\, dV = \oint_S (\hat{\mathbf{n}}\cdot\nabla)\boldsymbol{A}\, dS = \oint_S \frac{\partial \boldsymbol{A}}{\partial n}\, dS \tag{C.30}$$

$$\oint_S (\hat{\mathbf{n}}\times\nabla)\times\boldsymbol{A}\, dS = \oint_S \big[\hat{\mathbf{n}}\times(\nabla\times\boldsymbol{A}) + (\hat{\mathbf{n}}\cdot\nabla)\boldsymbol{A} - \hat{\mathbf{n}}(\nabla\cdot\boldsymbol{A})\big]\, dS = 0 \tag{C.31}$$

$$\int_V \nabla\times\boldsymbol{A}\, dV = \oint_S \hat{\mathbf{n}}\times\boldsymbol{A}\, dS \tag{C.32}$$

Using Eqs. (C.23) and (C.31), we find:

$$\begin{aligned}\oint_S \Big(\psi\,\frac{\partial \boldsymbol{E}}{\partial n} - \boldsymbol{E}\,\frac{\partial\psi}{\partial n}\Big)\, dS &= \\ = \oint_S \big[\hat{\mathbf{n}}\,\psi\,\nabla\cdot\boldsymbol{E} - (\hat{\mathbf{n}}\times\boldsymbol{E})\times\nabla\psi - \psi\,\hat{\mathbf{n}}\times(\nabla\times\boldsymbol{E}) &- (\hat{\mathbf{n}}\cdot\boldsymbol{E})\,\nabla\psi\big]\, dS\end{aligned} \tag{C.33}$$

The vectorial forms of Green's identities are [1116,1113]:

$$\int_V (\nabla\times\boldsymbol{A}\cdot\nabla\times\boldsymbol{B} - \boldsymbol{A}\cdot\nabla\times\nabla\times\boldsymbol{B})\, dV = \oint_S \hat{\mathbf{n}}\cdot(\boldsymbol{A}\times\nabla\times\boldsymbol{B})\, dS \tag{C.34}$$

$$\int_V (\boldsymbol{B}\cdot\nabla\times\nabla\times\boldsymbol{A} - \boldsymbol{A}\cdot\nabla\times\nabla\times\boldsymbol{B})\, dV = \oint_S \hat{\mathbf{n}}\cdot(\boldsymbol{A}\times\nabla\times\boldsymbol{B} - \boldsymbol{B}\times\nabla\times\boldsymbol{A})\, dS \tag{C.35}$$

Integral Theorems for Open Surfaces

Stokes' theorem involves an open surface S and its boundary contour C:

$$\boxed{\int_S \hat{\mathbf{n}}\cdot\nabla\times\boldsymbol{A}\, dS = \oint_C \boldsymbol{A}\cdot d\mathbf{l}} \qquad \text{(Stokes' theorem)} \tag{C.36}$$

where $d\mathbf{l}$ is the tangential path length around C. Some related theorems are:

$$\int_S \big[\psi\,\hat{\mathbf{n}}\cdot\nabla\times\boldsymbol{A} - (\hat{\mathbf{n}}\times\boldsymbol{A})\cdot\nabla\psi\big]\, dS = \oint_C \psi\boldsymbol{A}\cdot d\mathbf{l} \tag{C.37}$$

$$\int_S \big[(\nabla\psi)\,\hat{\mathbf{n}}\cdot\nabla\times\boldsymbol{A} - \big((\hat{\mathbf{n}}\times\boldsymbol{A})\cdot\nabla\big)\nabla\psi\big]\, dS = \oint_C (\nabla\psi)\boldsymbol{A}\cdot d\mathbf{l} \tag{C.38}$$

$$\int_S \hat{\mathbf{n}}\times\nabla\psi\, dS = \oint_C \psi\, d\mathbf{l} \tag{C.39}$$

$$\int_S (\hat{\mathbf{n}}\times\nabla)\times\boldsymbol{A}\, dS = \int_S \big[\hat{\mathbf{n}}\times(\nabla\times\boldsymbol{A}) + (\hat{\mathbf{n}}\cdot\nabla)\boldsymbol{A} - \hat{\mathbf{n}}(\nabla\cdot\boldsymbol{A})\big]\, dS = \oint_C d\mathbf{l}\times\boldsymbol{A} \tag{C.40}$$

$$\int_S \hat{\mathbf{n}}\, dS = \frac{1}{2}\oint_C \mathbf{r}\times d\mathbf{l} \tag{C.41}$$

Eq. (C.41) is a special case of (C.40). Using Eqs. (C.23) and (C.40) we find:

$$\begin{aligned}\int_S \Big(\psi\,\frac{\partial \boldsymbol{E}}{\partial n} - \boldsymbol{E}\,\frac{\partial\psi}{\partial n}\Big)\, dS + \oint_C \psi\boldsymbol{E}\times d\mathbf{l} &= \\ = \int_S \big[\hat{\mathbf{n}}\,\psi\,\nabla\cdot\boldsymbol{E} - (\hat{\mathbf{n}}\times\boldsymbol{E})\times\nabla\psi - \psi\,\hat{\mathbf{n}}\times(\nabla\times\boldsymbol{E}) &- (\hat{\mathbf{n}}\cdot\boldsymbol{E})\,\nabla\psi\big]\, dS\end{aligned} \tag{C.42}$$

D. Green's Functions

The Green's functions for the Laplace, Helmholtz, and one-dimensional Helmholtz equations are listed below:

$$\nabla^2 g(\mathbf{r}) = -\delta^{(3)}(\mathbf{r}) \quad\Rightarrow\quad g(\mathbf{r}) = \frac{1}{4\pi r} \tag{D.1}$$

$$(\nabla^2 + k^2)G(\mathbf{r}) = -\delta^{(3)}(\mathbf{r}) \quad\Rightarrow\quad G(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r} \tag{D.2}$$

$$(\partial_z^2 + \beta^2)g(z) = -\delta(z) \quad\Rightarrow\quad g(z) = \frac{e^{-j\beta|z|}}{2j\beta} \tag{D.3}$$

where $r = |\mathbf{r}|$. Eqs. (D.2) and (D.3) are appropriate for describing *outgoing* waves. We considered other versions of (D.3) in Sec. 21.3. A more general identity satisfied by the Green's function $g(\mathbf{r})$ of Eq. (D.1) is as follows (for a proof, see Refs. [111,112]):

$$\partial_i\partial_j g(\mathbf{r}) = -\frac{1}{3}\delta_{ij}\,\delta^{(3)}(\mathbf{r}) + \frac{3x_ix_j - r^2\delta_{ij}}{r^4}\, g(\mathbf{r}) \qquad i,j = 1,2,3 \tag{D.4}$$

where $\partial_i = \partial/\partial x_i$ and x_i stands for any of x,y,z. By summing the i,j indices, Eq. (D.4) reduces to (D.1). Using this identity, we find for the Green's function $G(\mathbf{r}) = e^{-jkr}/4\pi r$:

$$\partial_i\partial_j G(\mathbf{r}) = -\frac{1}{3}\delta_{ij}\,\delta^{(3)}(\mathbf{r}) + \Big[\Big(jk + \frac{1}{r}\Big)\frac{3x_ix_j - r^2\delta_{ij}}{r^3} - k^2\,\frac{x_ix_j}{r^2}\Big]G(\mathbf{r}) \tag{D.5}$$

This reduces to Eq. (D.2) upon summing the indices. For any fixed vector $\mathbf{p}$, Eq. (D.5) is equivalent to the vectorial identity:

$$\nabla\times\nabla\times\big[\mathbf{p}\,G(\mathbf{r})\big] = \frac{2}{3}\mathbf{p}\,\delta^{(3)}(\mathbf{r}) + \Big[\Big(jk + \frac{1}{r}\Big)\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}) - \mathbf{p}}{r^2} + k^2\,\hat{\mathbf{r}}\times(\mathbf{p}\times\hat{\mathbf{r}})\Big]G(\mathbf{r}) \tag{D.6}$$

The second term on the right is simply the left-hand side evaluated at points away from the origin, thus, we may write:

$$\nabla\times\nabla\times\big[\mathbf{p}\,G(\mathbf{r})\big] = \frac{2}{3}\mathbf{p}\,\delta^{(3)}(\mathbf{r}) + \Big[\nabla\times\nabla\times\big[\mathbf{p}\,G(\mathbf{r})\big]\Big]_{\mathbf{r}\neq 0} \tag{D.7}$$

Then, Eq. (D.7) implies the following integrated identity, where ∇ is with respect to $\mathbf{r}$:

$$\nabla\times\nabla\times\int_V \boldsymbol{P}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\,dV' = \frac{2}{3}\,\boldsymbol{P}(\mathbf{r})+\int_V\Big[\nabla\times\nabla\times\big[\boldsymbol{P}(\mathbf{r}')\,G(\mathbf{r}-\mathbf{r}')\big]\Big]_{\mathbf{r}'\neq\mathbf{r}}dV' \qquad \text{(D.8)}$$

and $\mathbf{r}$ is assumed to lie within V. If $\mathbf{r}$ is outside V, then the term $2\boldsymbol{P}(\mathbf{r})/3$ is absent.

Technically, the integrals in (D.8) are *principal-value* integrals, that is, the limits as $\delta\to 0$ of the integrals over $V-V_\delta(\mathbf{r})$, where $V_\delta(\mathbf{r})$ is an excluded small sphere of radius δ centered about $\mathbf{r}$. The $2\boldsymbol{P}(\mathbf{r})/3$ term has a different form if the excluded volume $V_\delta(\mathbf{r})$ has shape other than a sphere or a cube. See Refs. [1153,460,472,598] and [106–110] for the definitions and properties of such principal value integrals.

Another useful result is the so-called *Weyl representation* or plane-wave-spectrum representation [22,26,1153,27,515] of the outgoing Helmholtz Green's function $G(\mathbf{r})$:

$$\boxed{G(\mathbf{r})=\frac{e^{-jkr}}{4\pi r}=\int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty}\frac{e^{-j(k_xx+k_yy)}\,e^{-jk_z|z|}}{2jk_z}\,\frac{dk_x\,dk_y}{(2\pi)^2}} \qquad \text{(D.9)}$$

where $k_z^2 = k^2-k_\perp^2$, with $k_\perp=\sqrt{k_x^2+k_y^2}$. In order to correspond to either outgoing waves or decaying evanescent waves, k_z must be defined more precisely as follows:

$$k_z=\begin{cases}\sqrt{k^2-k_\perp^2}\,, & \text{if}\quad k_\perp\le k\,, \quad \text{(propagating modes)}\\ -j\sqrt{k_\perp^2-k^2}\,, & \text{if}\quad k_\perp> k\,, \quad \text{(evanescent modes)}\end{cases} \qquad \text{(D.10)}$$

The propagating modes are important in radiation problems and conventional imaging systems, such as Fourier optics [1156]. The evanescent modes are important in the new subject of *near-field optics*, in which objects can be probed and imaged at nanometer scales improving the resolution of optical microscopy by factors of ten. Some near-field optics references are [494–514].

To prove (D.9), we consider the two-dimensional spatial Fourier transform of $G(\mathbf{r})$ and its inverse. Indicating explicitly the dependence on the coordinates x,y,z, we have:

$$\begin{aligned} g(k_x,k_y,z) &= \int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} G(x,y,z)\,e^{j(k_xx+k_yy)}\,dx\,dy=\frac{e^{-jk_z|z|}}{2jk_z}\\ G(x,y,z) &= \int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty} g(k_x,k_y,z)\,e^{-j(k_xx+k_yy)}\,\frac{dk_x\,dk_y}{(2\pi)^2}\end{aligned} \qquad \text{(D.11)}$$

Writing $\delta^{(3)}(\mathbf{r})=\delta(x)\,\delta(y)\,\delta(z)$ and using the inverse Fourier transform:

$$\delta(x)\,\delta(y)=\int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty}e^{-j(k_xx+k_yy)}\,\frac{dk_x\,dk_y}{(2\pi)^2}\,,$$

we find from Eq. (D.2) that $g(k_x,k_y,z)$ must satisfy the one-dimensional Helmholtz Green's function equation (D.3), with $k_z^2=k^2-k_x^2-k_y^2=k^2-k_\perp^2$, that is,

$$(\partial_z^2+k_z^2)\,g(k_x,k_y,z)=-\delta(z) \qquad \text{(D.12)}$$

whose outgoing/evanescent solution is $g(k_x,k_y,z)=e^{-jk_z|z|}/2jk_z$.

A more direct proof of (D.9) is to use cylindrical coordinates, $k_x=k_\perp\cos\psi$, $k_y=k_\perp\sin\psi$, $x=\rho\cos\phi$, $y=\rho\sin\phi$, where $k_\perp^2=k_x^2+k_y^2$ and $\rho^2=x^2+y^2$. It follows that $k_xx+k_yy=k_\perp\rho\cos(\phi-\psi)$. Setting $dx\,dy=\rho\,d\rho\,d\phi=r\,dr\,d\phi$, the latter following from $r^2=\rho^2+z^2$, we obtain from Eq. (D.11) after replacing $\rho=\sqrt{r^2-z^2}$:

$$\begin{aligned} g(k_x,k_y,z) &= \iint\frac{e^{-jkr}}{4\pi r}\,e^{j(k_xx+k_yy)}\,dx\,dy=\iint\frac{e^{-jkr}}{4\pi r}\,e^{jk_\perp\rho\cos(\phi-\psi)}\,r\,dr\,d\phi\\ &=\frac{1}{2}\int_{|z|}^{\infty}dr\,e^{-jkr}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{jk_\perp\rho\cos(\phi-\psi)}=\frac{1}{2}\int_{|z|}^{\infty}dr\,e^{-jkr}\,J_0\big(k_\perp\sqrt{r^2-z^2}\big)\end{aligned}$$

where we used the integral representation (17.9.2) of the Bessel function $J_0(x)$. Looking up the last integral in the table of integrals [1268], we find:

$$\boxed{g(k_x,k_y,z)=\frac{1}{2}\int_{|z|}^{\infty}dr\,e^{-jkr}\,J_0\big(k_\perp\sqrt{r^2-z^2}\big)=\frac{e^{-jk_z|z|}}{2jk_z}} \qquad \text{(D.13)}$$

where k_z must be defined exactly as in Eq. (D.10). A direct consequence of Eq. (D.11) and the even-ness of $G(\mathbf{r})$ in $\mathbf{r}$ and of $g(k_x,k_y,z)$ in k_x,k_y, is the following result:

$$\int_{-\infty}^{\infty}\!\int_{-\infty}^{\infty}e^{-j(k_xx'+k_yy')}\,G(\mathbf{r}-\mathbf{r}')\,dx'dy'=e^{-j(k_xx+k_yy)}\,\frac{e^{-jk_z|z-z'|}}{2jk_z} \qquad \text{(D.14)}$$

One can also show the integral:

$$\int_0^{\infty}e^{-jk_z'z'}\,\frac{e^{-jk_z|z-z'|}}{2jk_z}\,dz'=\begin{cases}\dfrac{e^{-jk_z'z}}{k_z'^2-k_z^2}-\dfrac{e^{-jk_zz}}{2k_z(k_z'-k_z)}\,, & \text{for}\quad z\ge 0\\[2ex] -\dfrac{e^{jk_zz}}{2k_z(k_z'+k_z)}\,, & \text{for}\quad z<0\end{cases} \qquad \text{(D.15)}$$

The proof is obtained by splitting the integral over the sub-intervals $[0,z]$ and $[z,\infty)$. To handle the limits at infinity, k_z' must be assumed to be slightly lossy, that is, $k_z'=\beta_z-j\alpha_z$, with $\alpha_z>0$. Eqs. (D.14) and (D.15) can be combined into:

$$\int_{V_+}e^{-j\,\boldsymbol{k}'\cdot\mathbf{r}'}\,G(\mathbf{r}-\mathbf{r}')\,dV'=\begin{cases}\dfrac{e^{-j\,\boldsymbol{k}'\cdot\mathbf{r}}}{k'^2-k^2}-\dfrac{e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}}{2k_z(k_z'-k_z)}\,, & \text{for}\quad z\ge 0\\[2ex] -\dfrac{e^{-j\,\boldsymbol{k}_-\cdot\mathbf{r}}}{2k_z(k_z'+k_z)}\,, & \text{for}\quad z<0\end{cases} \qquad \text{(D.16)}$$

where V_+ is the half-space $z\ge 0$, and $\boldsymbol{k}$, $\boldsymbol{k}_-$, $\boldsymbol{k}'$ are wave-vectors with the same k_x,k_y components, but different k_zs:

$$\begin{aligned}\boldsymbol{k} &= k_x\,\hat{\mathbf{x}}+k_y\,\hat{\mathbf{y}}+k_z\,\hat{\mathbf{z}}\\ \boldsymbol{k}_- &= k_x\,\hat{\mathbf{x}}+k_y\,\hat{\mathbf{y}}-k_z\,\hat{\mathbf{z}}\\ \boldsymbol{k}' &= k_x\,\hat{\mathbf{x}}+k_y\,\hat{\mathbf{y}}+k_z'\,\hat{\mathbf{z}}\end{aligned} \qquad \text{(D.17)}$$

where we note that $k'^2-k^2=(k_x^2+k_y^2+k_z'^2)-(k_x^2+k_y^2+k_z^2)=k_z'^2-k_z^2$.

The Green's function results (D.8)–(D.17) are used in the discussion of the Ewald-Oseen extinction theorem in Sec. 14.6.

A related Weyl-type representation is obtained by differentiating Eq. (D.9) with respect to z. Assuming that $z \ge 0$ and interchanging differentiation and integration (and multiplying by -2), we obtain the identity:

$$\boxed{-2\frac{\partial}{\partial z}\left(\frac{e^{-jkr}}{4\pi r}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-jk_x x}\, e^{-jk_y y} e^{-jk_z z}\,\frac{dk_x\,dk_y}{(2\pi)^2}}\,,\quad z \ge 0 \tag{D.18}$$

This just means that the left-hand side is the two-dimensional inverse Fourier transform of $e^{-jk_z z}$ with k_z given by Eq. (D.10). Replacing $\mathbf{r}$ by $\mathbf{r}-\mathbf{r}'$, and r by $R = |\mathbf{r}-\mathbf{r}'|$, and noting that $\partial_{z'} = -\partial_z$, we also obtain:

$$\boxed{2\frac{\partial}{\partial z'}\left(\frac{e^{-jkR}}{4\pi R}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-jk_x(x-x')}\, e^{-jk_y(y-y')} e^{-jk_z(z-z')}\,\frac{dk_x\,dk_y}{(2\pi)^2}}\,,\quad z \ge z' \tag{D.19}$$

This result establishes the equivalence between the Kirchhoff-Fresnel diffraction formula and the plane-wave spectrum representation as discussed in Sec. 17.17. For the vector diffraction case, we also need the derivatives of G with respect to the transverse coordinates x, y. Differentiating (D.9) with respect to x (or with respect to y), we have:

$$\boxed{-2\frac{\partial}{\partial x}\left(\frac{e^{-jkr}}{4\pi r}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k_x}{k_z}\, e^{-jk_x x}\, e^{-jk_y y} e^{-jk_z z}\,\frac{dk_x\,dk_y}{(2\pi)^2}}\,,\quad z \ge 0 \tag{D.20}$$

E. Coordinate Systems

The definitions of cylindrical and spherical coordinates were given in Sec. 14.8. The expressions of the gradient, divergence, curl, Laplacian operators, and delta functions are given below in cartesian, cylindrical, and spherical coordinates.

Cartesian Coordinates

$$\begin{aligned}
\nabla\psi &= \hat{\mathbf{x}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z} \\
\nabla^2\psi &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \\
\nabla\cdot\boldsymbol{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla\times\boldsymbol{A} &= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial_x} & \dfrac{\partial}{\partial_y} & \dfrac{\partial}{\partial_z} \\ A_x & A_y & A_z \end{vmatrix} \\
\delta^{(3)}(\mathbf{r}-\mathbf{r}') &= \delta(x-x')\,\delta(y-y')\,\delta(z-z')
\end{aligned} \tag{E.1}$$

Cylindrical Coordinates

$$\nabla\psi = \hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z} \tag{E.2a}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{E.2b}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{E.2c}$$

$$\nabla\times\boldsymbol{A} = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\mathbf{z}}\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right) \tag{E.2d}$$

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi')\,\delta(z-z') \tag{E.2e}$$

Spherical Coordinates

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{E.3a}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{E.3b}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{E.3c}$$

$$\begin{aligned}\nabla\times\boldsymbol{A} = \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\\ + \hat{\boldsymbol{\phi}}\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\end{aligned} \tag{E.3d}$$

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \frac{1}{r^2\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi') \tag{E.3e}$$

Transformations Between Coordinate Systems

A vector $\boldsymbol{A}$ can be expressed component-wise in the three coordinate systems as:

$$\begin{aligned}
\boldsymbol{A} &= \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z \\
&= \hat{\boldsymbol{\rho}}A_\rho + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z \\
&= \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi
\end{aligned} \tag{E.4}$$

The components in one coordinate system can be expressed in terms of the components of another by using the following relationships between the unit vectors, which were also given in Eqs. (14.8.1)–(14.8.3):

$$\begin{aligned} x &= \rho\cos\phi & \hat{\boldsymbol{\rho}} &= \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi & \hat{\mathbf{x}} &= \hat{\boldsymbol{\rho}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \\ y &= \rho\sin\phi & \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi & \hat{\mathbf{y}} &= \hat{\boldsymbol{\rho}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \end{aligned} \tag{E.5}$$

$$\begin{aligned} \rho &= r\sin\theta & \hat{\mathbf{r}} &= \hat{\mathbf{z}}\cos\theta + \hat{\boldsymbol{\rho}}\sin\theta & \hat{\mathbf{z}} &= \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \\ z &= r\cos\theta & \hat{\boldsymbol{\theta}} &= -\hat{\mathbf{z}}\sin\theta + \hat{\boldsymbol{\rho}}\cos\theta & \hat{\boldsymbol{\rho}} &= \hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta \end{aligned} \tag{E.6}$$

$$\begin{aligned} x &= r\sin\theta\cos\phi & \hat{\mathbf{r}} &= \hat{\mathbf{x}}\cos\phi\sin\theta + \hat{\mathbf{y}}\sin\phi\sin\theta + \hat{\mathbf{z}}\cos\theta \\ y &= r\sin\theta\sin\phi & \hat{\boldsymbol{\theta}} &= \hat{\mathbf{x}}\cos\phi\cos\theta + \hat{\mathbf{y}}\sin\phi\cos\theta - \hat{\mathbf{z}}\sin\theta \\ z &= r\cos\theta & \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \end{aligned} \tag{E.7}$$

$$\begin{aligned} \hat{\mathbf{x}} &= \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \\ \hat{\mathbf{y}} &= \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \\ \hat{\mathbf{z}} &= \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \end{aligned} \tag{E.8}$$

For example, to express the spherical components A_θ, A_ϕ in terms of the cartesian components, we proceed as follows:

$$A_\theta = \hat{\boldsymbol{\theta}}\cdot\mathbf{A} = \hat{\boldsymbol{\theta}}\cdot(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) = (\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{x}})\,A_x + (\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{y}})\,A_y + (\hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{z}})\,A_z$$

$$A_\phi = \hat{\boldsymbol{\phi}}\cdot\mathbf{A} = \hat{\boldsymbol{\phi}}\cdot(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) = (\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{x}})\,A_x + (\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{y}})\,A_y + (\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}})\,A_z$$

The dot products can be read off Eq. (E.7), resulting in:

$$\begin{aligned} A_\theta &= \cos\phi\cos\theta A_x + \sin\phi\cos\theta A_y - \sin\theta A_z \\ A_\phi &= -\sin\phi A_x + \cos\phi A_y \end{aligned} \tag{E.9}$$

Similarly, using Eq. (E.6) the cylindrical components A_ρ, A_z can be expressed in terms of spherical components as:

$$\begin{aligned} A_\rho &= \hat{\boldsymbol{\rho}}\cdot\mathbf{A} = \hat{\boldsymbol{\rho}}\cdot(\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi) = \sin\theta A_r + \cos\theta A_\theta \\ A_z &= \hat{\mathbf{z}}\cdot\mathbf{A} = \hat{\mathbf{z}}\cdot(\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi) = \cos\theta A_r - \cos\theta A_\theta \end{aligned} \tag{E.10}$$

F. Fresnel, Exponential, Sine, and Cosine Integrals

The Fresnel functions $C(x)$ and $S(x)$ are defined by [1267]:

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\,, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt \tag{F.1}$$

They may be combined into the complex function:

$$\boxed{\mathcal{F}(x) = C(x) - jS(x) = \int_0^x e^{-j(\pi/2)t^2}\,dt} \tag{F.2}$$

$C(x)$, $S(x)$, and $\mathcal{F}(x)$ are *odd* functions of x and have the asymptotic values:

$$C(\infty) = S(\infty) = \frac{1}{2}\,, \quad \mathcal{F}(\infty) = \frac{1-j}{2} \tag{F.3}$$

At $x = 0$, we have $\mathcal{F}(0) = 0$ and $\mathcal{F}'(0) = 1$, so that the Taylor series approximation is $\mathcal{F}(x) \simeq x$, for small x. The asymptotic expansions of $C(x)$, $S(x)$, and $\mathcal{F}(x)$ are for large positive x:

$$\begin{aligned} \mathcal{F}(x) &= \frac{1-j}{2} + \frac{j}{\pi x}\,e^{-j\pi x^2/2} \\ C(x) &= \frac{1}{2} + \frac{1}{\pi x}\,\sin\left(\frac{\pi}{2}x^2\right) \\ S(x) &= \frac{1}{2} - \frac{1}{\pi x}\,\cos\left(\frac{\pi}{2}x^2\right) \end{aligned} \tag{F.4}$$

Associated with $C(x)$ and $S(x)$ are the type-2 Fresnel integrals:

$$C_2(x) = \int_0^x \frac{\cos t}{\sqrt{2\pi t}}\,dt\,, \quad S_2(x) = \int_0^x \frac{\sin t}{\sqrt{2\pi t}}\,dt \tag{F.5}$$

They are combined into the complex function:

$$\mathcal{F}_2(x) = C_2(x) - jS_2(x) = \int_0^x \frac{e^{-jt}}{\sqrt{2\pi t}}\,dt \tag{F.6}$$

The two types are related by, if $x \geq 0$:

$$C(x) = C_2\left(\frac{\pi}{2}x^2\right), \quad S(x) = S_2\left(\frac{\pi}{2}x^2\right), \quad \mathcal{F}(x) = \mathcal{F}_2\left(\frac{\pi}{2}x^2\right) \tag{F.7}$$

and if $x < 0$, we set $\mathcal{F}(x) = -\mathcal{F}(-x) = -\mathcal{F}_2(\pi x^2/2)$.

The Fresnel function $\mathcal{F}_2(x)$ can be evaluated numerically using Boersma's approximation [1132], which achieves a maximum error of 10^{-9} over all x. The algorithm approximates the function $\mathcal{F}_2(x)$ as follows:

$$\mathcal{F}_2(x) = \begin{cases} e^{-jx}\sqrt{\dfrac{x}{4}}\displaystyle\sum_{n=0}^{11}(a_n + jb_n)\left(\frac{x}{4}\right)^n, & \text{if} \quad 0 \leq x \leq 4 \\ \dfrac{1-j}{2} + e^{-jx}\sqrt{\dfrac{4}{x}}\displaystyle\sum_{n=0}^{11}(c_n + jd_n)\left(\frac{4}{x}\right)^n, & \text{if} \quad x > 4 \end{cases} \tag{F.8}$$

where the coefficients a_n, b_n, c_n, d_n are given in [1132]. Consistency with the small- and large-x expansions of $\mathcal{F}(x)$ requires that $a_0 + jb_0 = \sqrt{8/\pi}$ and $c_0 + jd_0 = j/\sqrt{8\pi}$. We have implemented Eq. (F.8) with the MATLAB function `fcs2`:

```
F2 = fcs2(x);          % Fresnel integrals F2(x) = C2(x) - jS2(x)
```

The ordinary Fresnel integral $\mathcal{F}(x)$ can be computed with the help of Eq. (F.7). The MATLAB function `fcs` calculates $\mathcal{F}(x)$ for any vector of values x by calling `fcs2`:

```
F = fcs(x);          % Fresnel integrals F(x) = C(x) - jS(x)
```

In calculating the radiation patterns of pyramidal horns, it is desired to calculate a Fresnel diffraction integral of the type:

$$\boxed{F_0(\nu,\sigma) = \int_{-1}^{1} e^{j\pi\nu\xi}\,e^{-j(\pi/2)\sigma^2\xi^2}\,d\xi} \tag{F.9}$$

Making the variable change $t = \sigma\xi - \nu/\sigma$, this integral can be computed in terms of the Fresnel function $\mathcal{F}(x) = C(x) - jS(x)$ as follows:

$$\boxed{F_0(\nu,\sigma) = \frac{1}{\sigma}\, e^{j(\pi/2)(\nu^2/\sigma^2)} \left[\mathcal{F}\left(\frac{\nu}{\sigma}+\sigma\right) - \mathcal{F}\left(\frac{\nu}{\sigma}-\sigma\right)\right]} \tag{F.10}$$

where we also used the oddness of $\mathcal{F}(x)$. The value of Eq. (F.9) at $\nu = 0$ is:

$$\boxed{F_0(0,\sigma) = \frac{1}{\sigma}\left[\mathcal{F}(\sigma) - \mathcal{F}(-\sigma)\right] = 2\,\frac{\mathcal{F}(\sigma)}{\sigma}} \tag{F.11}$$

Eq. (F.10) assumes that $\sigma \neq 0$. If $\sigma = 0$, the integral (F.9) reduces to the sinc function:

$$\boxed{F_0(\nu,0) = 2\,\frac{\sin(\pi\nu)}{\pi\nu}} \tag{F.12}$$

From either (F.11) or (F.12), we find $F_0(0,0) = 2$. A related integral that is also required in the theory of horns is the following:

$$\boxed{F_1(\nu,\sigma) = \int_{-1}^{1} \cos\left(\frac{\pi\xi}{2}\right) e^{j\pi\nu\xi}\, e^{-j(\pi/2)\sigma^2\xi^2}\, d\xi} \tag{F.13}$$

Writing $\cos(\pi\xi/2) = (e^{j\pi\xi/2} + e^{-j\pi\xi/2})/2$, the integral $F_1(\nu, s)$ can be expressed in terms of $F_0(\nu,\sigma)$ as follows:

$$\boxed{F_1(\nu,\sigma) = \frac{1}{2}\left[F_0(\nu+0.5,\sigma) + F_0(\nu-0.5,\sigma)\right]} \tag{F.14}$$

It can be verified easily that $F_0(0.5,\sigma) = F_0(-0.5,\sigma)$, therefore, the value of $F_1(\nu,\sigma)$ at $\nu = 0$ will be given by:

$$\boxed{F_1(0,\sigma) = F_0(0.5,\sigma) = \frac{1}{\sigma}\, e^{j\pi/(8\sigma^2)} \left[\mathcal{F}\left(\frac{1}{2\sigma}+\sigma\right) - \mathcal{F}\left(\frac{1}{2\sigma}-\sigma\right)\right]} \tag{F.15}$$

Using the asymptotic expansion (F.4), we find the expansion valid for small σ:

$$\mathcal{F}\left(\frac{1}{2\sigma} \pm \sigma\right) = \frac{1-j}{2} \mp \frac{2\sigma}{\pi}\, e^{-j\pi/(8\sigma^2)}, \quad \text{for small } \sigma \tag{F.16}$$

For $\sigma = 0$, the integral $F_1(\nu,\sigma)$ reduces to the double-sinc function:

$$\begin{aligned} F_1(\nu,0) &= \int_{-1}^{1} \cos\left(\frac{\pi\xi}{2}\right) e^{j\pi\nu\xi}\, d\xi = \frac{1}{2}\left[F_0(\nu+0.5,0) + F_0(\nu-0.5,0)\right] \\ &= \frac{\sin\big(\pi(\nu+0.5)\big)}{\pi(\nu+0.5)} + \frac{\sin\big(\pi(\nu-0.5)\big)}{\pi(\nu-0.5)} = \frac{4}{\pi}\,\frac{\cos(\pi\nu)}{1-4\nu^2} \end{aligned} \tag{F.17}$$

From either Eq. (F.16) or (F.17), we find $F_1(0,0) = 4/\pi$.

The MATLAB function `diffint` can be used to evaluate both Eq. (F.9) and (F.13) for any vector of values ν and any vector of positive numbers σ, including $\sigma = 0$. It calls `fcs` to evaluate the diffraction integral (F.9) according to Eq. (F.10). Its usage is:


```
F0 = diffint(v,sigma,0);          % diffraction integral F0(v,σ), Eq. (F.9)
F1 = diffint(v,sigma,1);          % diffraction integral F1(v,σ), Eq. (F.13)
```

The vectors `v,sigma` can be entered either as rows or columns, but the result will be a matrix of size `length(v) x length(sigma)`. The integral $F_0(\nu,\sigma)$ can also be calculated by the simplified call:

```
F0 = diffint(v,sigma);            % diffraction integral F0(v,σ), Eq. (F.9)
```

Actually, the most general syntax of `diffint` is as follows:

```
F = diffint(v,sigma,a,c1,c2);     % diffraction integral F(v,σ,a), Eq. (F.18)
```

It evaluates the more general integral:

$$\boxed{F(\nu,\sigma,a) = \int_{c_1}^{c_2} \cos\left(\frac{\pi\xi a}{2}\right) e^{j\pi\nu\xi}\, e^{-j(\pi/2)\sigma^2\xi^2}\, d\xi} \tag{F.18}$$

For $a = 0$, we have:

$$\boxed{F(\nu,\sigma,0) = \frac{1}{\sigma}\, e^{j(\pi/2)(\nu^2/\sigma^2)} \left[\mathcal{F}\left(\frac{\nu}{\sigma}-\sigma c_1\right) - \mathcal{F}\left(\frac{\nu}{\sigma}-\sigma c_2\right)\right]} \tag{F.19}$$

For $a \neq 0$, we can express $F(\nu,\sigma,a)$ in terms of $F(\nu,\sigma,0)$:

$$\boxed{F(\nu,\sigma,a) = \frac{1}{2}\left[F(\nu+0.5a,\sigma,0) + F(\nu-0.5a,\sigma,0)\right]} \tag{F.20}$$

For $a = 0$ and $\sigma = 0$, $F(\nu,\sigma,a)$ reduces to the complex sinc function:

$$\boxed{F(\nu,0,0) = \frac{e^{j\pi\nu c_2} - e^{j\pi\nu c_1}}{j\pi\nu} = (c_2-c_1)\,\frac{\sin\big(\pi(c_2-c_1)\nu/2\big)}{\pi(c_2-c_1)\nu/2}\, e^{j\pi(c_2+c_1)\nu/2}} \tag{F.21}$$

Stationary Phase Approximation

The Fresnel integrals find also application in the the stationary-phase approximation for evaluating integrals. The approximation can be stated as follows:

$$\boxed{\int_{-\infty}^{\infty} f(x)\, e^{j\phi(x)}\, dx \simeq \sqrt{\frac{2\pi j}{\phi''(x_0)}}\, f(x_0)\, e^{j\phi(x_0)}} \tag{F.22}$$

where x_0 is a stationary point of the phase $\phi(x)$, that is, the solution of $\phi'(x_0) = 0$, where for simplicity we assume that there is only one such point (otherwise, one has a sum of terms like (F.22), one for each solution of $\phi'(x) = 0$). Eq. (F.22) is obtained by expanding $\phi(x)$ in Taylor series about the stationary point $x = x_0$ and keeping only up to the quadratic term:

$$\phi(x) \simeq \phi(x_0) + \phi'(x_0)(x-x_0) + \frac{1}{2}\phi''(x_0)(x-x_0)^2 = \phi(x_0) + \frac{1}{2}\phi''(x_0)(x-x_0)^2$$

Making this approximation in the integral and assuming that $f(x)$ is slowly varying in the neighborhood of x_0, we may replace $f(x)$ by its value at x_0:

$$\int_{-\infty}^{\infty} f(x)\,e^{j\phi(x)}\,dx \simeq \int_{-\infty}^{\infty} f(x_0)\,e^{j\left(\phi(x_0)+\phi''(x_0)(x-x_0)^2/2\right)}\,dx$$
$$= f(x_0)\,e^{j\phi(x_0)}\int_{-\infty}^{\infty} e^{j\phi''(x_0)(x-x_0)^2/2}dx$$

The last integral can be reduced to the complex Fresnel integral by the change of variables $(x-x_0)=\sqrt{\pi/\phi''(x_0)}\,u$:

$$\int_{-\infty}^{\infty} e^{j\phi''(x_0)(x-x_0)^2/2}dx = \sqrt{\frac{\pi}{\phi''(x_0)}}\int_{-\infty}^{\infty} e^{j\pi u^2/2}du = \sqrt{\frac{\pi}{\phi''(x_0)}}\left[\mathcal{F}(\infty)-\mathcal{F}(-\infty)\right]^*$$

Using $\left[\mathcal{F}(\infty)-\mathcal{F}(-\infty)\right]^* = 2\mathcal{F}^*(\infty)=1+j=\sqrt{2j}$, we obtain

$$\int_{-\infty}^{\infty} e^{j\phi''(x_0)(x-x_0)^2/2}dx = \sqrt{\frac{2\pi j}{\phi''(x_0)}}$$

Normally, the phase depends on a positive parameter λ in the form $\phi(x)=\lambda\theta(x)$, and the stationary-phase approximation is justified in the limit $\lambda\to\infty$.

Exponential, Sine, and Cosine Integrals

Several antenna calculations, such as mutual impedances and directivities, can be reduced to the exponential integral, which is defined as follows [1267]:

$$E_1(z)=\int_z^{\infty}\frac{e^{-u}}{u}\,du = e^{-z}\int_0^{\infty}\frac{e^{-t}}{z+t}\,dt \qquad \text{(exponential integral)} \tag{F.23}$$

where z is a complex number with phase restricted such that $|\arg z|<\pi$. This range allows pure imaginary z's. The built-in MATLAB function `expint` evaluates $E_1(z)$ at an array of z's. Related to $E_1(z)$ are the sine and cosine integrals:

$$\begin{aligned} S_i(z)&=\int_0^z\frac{\sin u}{u}\,du && \text{(sine integral)}\\ C_i(z)&=\gamma+\ln z+\int_0^z\frac{\cos u-1}{u}\,du && \text{(cosine integral)}\end{aligned} \tag{F.24}$$

where γ is the Euler constant $\gamma = 0.5772156649\dots$. A related cosine integral is:

$$C_{\rm in}(z)=\int_0^z\frac{1-\cos u}{u}\,du = \gamma+\ln z - C_i(z) \tag{F.25}$$

For $z\ge 0$, the sine and cosine integrals are related to $E_1(z)$ by [1267]:

$$\begin{aligned} S_i(z)&=\frac{E_1(jz)-E_1(-jz)}{2j}+\frac{\pi}{2} = \mathrm{Im}\left[E_1(jz)\right]+\frac{\pi}{2}\\ C_i(z)&=-\frac{E_1(jz)+E_1(-jz)}{2} = -\mathrm{Re}\left[E_1(jz)\right]\end{aligned} \tag{F.26}$$

while for $z\le 0$, we have $S_i(z)=-S_i(-z)$ and $C_i(z)=C_i(-z)+j\pi$. Conversely, we have for $z>0$:

$$E_1(jz)=-C_i(z)+j\big(S_i(z)-\frac{\pi}{2}\big) = -\gamma-\ln(z)+C_{\rm in}(z)+j\big(S_i(z)-\frac{\pi}{2}\big) \tag{F.27}$$

The MATLAB functions `Si`, `Ci`, `Cin` evaluate the sine and cosine integrals at any vector of z's by using the relations (F.26) and the built-in function `expint`:

```
y = Si(z);      % sine integral, Eq. (F.24)
y = Ci(z);      % sine integral, Eq. (F.24)
y = Cin(z);     % sine integral, Eq. (F.25)
```

A related integral that appears in calculating mutual and self impedances is what may be called a "Green's function integral":

$$\mathrm{Gi}(d,z_0,h,s)=\int_0^h\frac{e^{-jkR}}{R}\,e^{-jksz}\,dz\,,\quad R=\sqrt{d^2+(z-z_0)^2}\,,\quad s=\pm1 \tag{F.28}$$

This integral can be reduced to the exponential integral by the change of variables:

$$v = jk\big(R+s(z-z_0)\big) \quad\Rightarrow\quad s\,\frac{dv}{v}=\frac{dz}{R}$$

which gives

$$\int_0^h\frac{e^{-jkR}}{R}\,e^{-jksz}\,dz = se^{-jksz_0}\int_{v_0}^{v_1}\frac{e^{-u}}{u}\,du\,,\quad\text{or,}$$

$$\boxed{\mathrm{Gi}(d,z_0,h,s)=\int_0^h\frac{e^{-jkR}}{R}\,e^{-jksz}\,dz = se^{-jksz_0}\left[E_1(ju_0)-E_1(ju_1)\right]} \tag{F.29}$$

where

$$\begin{aligned} v_0&=ju_0\,,\quad u_0=k\left[\sqrt{d^2+z_0^2}-sz_0\right]\\ v_1&=ju_1\,,\quad u_1=k\left[\sqrt{d^2+(h-z_0)^2}+s(h-z_0)\right]\end{aligned}$$

The function `Gi` evaluates Eq. (F.29), where z_0, s, and the resulting integral J, can be vectors of the same dimension. Its usage is:

```
J = Gi(d,z0,h,s);       % Green's function integral, Eq. (F.29)
```

Another integral that appears commonly in antenna work is:

$$\int_0^{\pi}\frac{\cos(\alpha\cos\theta)-\cos\alpha}{\sin\theta}\,d\theta = S_i(2\alpha)\sin\alpha - C_{\rm in}(2\alpha)\cos\alpha \tag{F.30}$$

Its proof is straightforward by first changing variables to $z=\cos\theta$, then using partial fraction expansion, and finally changing variables to $u=\alpha(1+z)$, and using the definitions (F.24) and (F.25):

$$\int_0^{\pi}\frac{\cos(\alpha\cos\theta)-\cos\alpha}{\sin\theta}\,d\theta = \int_{-1}^{1}\frac{\cos(\alpha z)-\cos\alpha}{1-z^2}\,dz$$
$$=\frac{1}{2}\int_{-1}^{1}\frac{\cos(\alpha z)-\cos\alpha}{1+z}\,dz+\frac{1}{2}\int_{-1}^{1}\frac{\cos(\alpha z)-\cos\alpha}{1-z}\,dz = \int_{-1}^{1}\frac{\cos(\alpha z)-\cos\alpha}{1+z}\,dz$$
$$=\int_0^{2\alpha}\frac{\cos(u-\alpha)-\cos\alpha}{u}\,du = \sin\alpha\int_0^{2\alpha}\frac{\sin u}{u}\,du-\cos\alpha\int_0^{2\alpha}\frac{1-\cos u}{u}\,du$$

G. Gauss-Legendre Quadrature

In many parts of this book it is necessary to perform numerical integration. Gauss-Legendre quadrature is one of the best integration methods, and we have implemented it with the MATLAB functions `quadr` and `quadrs`. Below, we give a brief description of the method.† The integral over an interval $[a, b]$ is approximated by a sum of the form:

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^{N} w_i f(x_i) \tag{G.1}$$

where w_i, x_i are appropriate weights and evaluation points (nodes). This can be written in the vectorial form:

$$\int_a^b f(x)\,dx \simeq \sum_{i=1}^{N} w_i f(x_i) = [w_1, w_2, \dots, w_N]\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} = \mathbf{w}^T f(\mathbf{x}) \tag{G.2}$$

The function `quadr` returns the column vectors of weights $\mathbf{w}$ and nodes $\mathbf{x}$, with usage:

```
[w,x] = quadr(a,b,N);      Gauss-Legendre quadrature
```

The function `quadrs` allows the splitting of the interval $[a, b]$ into subintervals, computes N weights and nodes in each subinterval, and concatenates them to form the overall weight and node vectors $\mathbf{w}, \mathbf{x}$:

```
[w,x] = quadrs(ab,N);      Gauss-Legendre quadrature over subintervals
```

where `ab` is an array of endpoints that define the subintervals, for example,

`ab` $= [a, b]$,	single interval
`ab` $= [a, c, b]$,	two subintervals, $[a, c]$ and $[c, b]$
`ab` $= [a, c, d, b]$,	three subintervals, $[a, c]$, $[c, d]$, and $[d, b]$
`ab` $= a : c : b$,	subintervals, $[a,\ a{+}c,\ a{+}2c,\ \dots,\ a{+}Mc]$, with $a + Mc = b$

As an example, consider the following function and its exact integral:

$$f(x) = e^x + \frac{1}{x}, \qquad J = \int_1^2 f(x)\,dx = e^2 - e^1 + \ln 2 = 5.36392145$$

This integral can be evaluated numerically by the MATLAB code:

```
N = 5;                     % number of weights and nodes
[w,x] = quadr(1,2,N);      % calculate weights and nodes for the interval [1,2]
f = exp(x) + 1./x;         % evaluate f(x) at the node vector
J = w'*f                   % approximate integral
```

This produces the exact value with a 4.23×10^{-7} percentage error. If the integration interval is split in two, say, $[1, 1.5]$ and $[1.5, 2]$, then the second line above can be replaced by

† J. Stoer and R. Burlisch, *Introduction to Numerical Analysis*, Springer, NY, (1980); and, G. H. Golub and J. H. Welsch, "Calculation of Gauss Quadrature Rules," *Math. Comput.*, **23**, 221 (1969).


```
[w,x] = quadrs([1,1.5,2],N);      % or by, [w,x] = quadrs(1:0.5:2, N);
```

which has a percentage error of 1.28×10^{-9}. Next, we discuss the theoretical basis of the method.

The interval $[a, b]$ can be replaced by the standardized interval $[-1, 1]$ with the transformation from $a \le x \le b$ to $-1 \le z \le 1$:

$$x = \left(\frac{b-a}{2}\right) z + \left(\frac{b+a}{2}\right) \tag{G.3}$$

If w_i and z_i are the weights and nodes with respect to the interval $[-1, 1]$, then those with respect to $[a, b]$ can be constructed simply as follows, for $i = 1, 2, \dots, N$:

$$\begin{aligned} x_i &= \left(\frac{b-a}{2}\right) z_i + \left(\frac{b+a}{2}\right) \\ w_i^x &= \left(\frac{b-a}{2}\right) w_i \end{aligned} \tag{G.4}$$

where the scaling of the weights follows from the scaling of the differentials $dx = dz(b-a)/2$, so the value of the integral (G.1) is preserved by the transformation.

Gauss-Legendre quadrature is nicely tied with the theory of orthogonal polynomials over the interval $[-1, 1]$, which are the Legendre polynomials. For N-point quadrature, the nodes z_i, $i = 1, 2, \dots, N$ are the N roots of the Legendre polynomial $P_N(z)$, which all lie in the interval $[-1, 1]$. The method is justified by the following theorem:

For any polynomial $P(z)$ of degree at most $2N - 1$, the quadrature formula (G.1) is satisfied *exactly*, that is,

$$\int_{-1}^{1} P(z)\,dz = \sum_{i=1}^{N} w_i P(z_i) \tag{G.5}$$

provided that the z_i are the N roots of the Legendre polynomial $P_N(z)$.

The Legendre polynomials $P_n(z)$ are obtained via the process of Gram-Schmidt orthogonalization of the non-orthogonal monomial basis $\{1, z, z^2, \dots, z^n \dots\}$. Orthogonality is defined with respect to the following inner product over the interval $[-1, 1]$:

$$(f, g) = \int_{-1}^{1} f(z) g(z)\,dz \tag{G.6}$$

The standard definition of the Legendre polynomials is:

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left[(z^2-1)^n\right], \quad n = 0, 1, 2, \dots \tag{G.7}$$

The first few of them are listed below:

$$\begin{aligned} P_0(z) &= 1 \\ P_1(z) &= z \\ P_2(z) &= (3/2)\left[z^2 - (1/3)\right] \\ P_3(z) &= (5/2)\left[z^3 - (3/5)z\right] \\ P_4(z) &= (35/8)\left[z^4 - (6/7)z^2 + (3/35)\right] \end{aligned} \tag{G.8}$$

They are normalized such that $P_n(1)=1$ and are mutually orthogonal with respect to (G.6), but do not have unit norm:

$$(P_n,P_m)=\int_{-1}^{1}P_n(z)P_m(z)\,dz=\frac{2}{2n+1}\,\delta_{nm} \tag{G.9}$$

Moreover, they satisfy the three-term recurrence relation:

$$zP_n(z)=\left(\frac{n}{2n+1}\right)P_{n-1}(z)+\left(\frac{n+1}{2n+1}\right)P_{n+1}(z) \tag{G.10}$$

The Gram-Schmidt orthogonalization process of the monomial basis $f_n(z)=z^n$ is the following order-recursive construction:

$$\begin{aligned}&\text{initialize}\quad P_0(z)=f_0(z)=1\\&\text{for } n=1,2,3,\dots,\ \text{do}\\&\qquad P_n(z)=f_n(z)-\sum_{k=0}^{n-1}\frac{(f_n,P_k)}{(P_k,P_k)}P_k(z)\end{aligned}$$

A few steps of the construction will clarify it:

$$P_1(z)=f_1(z)-\frac{(f_1,P_0)}{(P_0,P_0)}P_0(z)=z$$

where $(f_1,P_0)=(z,1)=\int_{-1}^{1}z\,dz=0$. Then, construct P_2 by:

$$P_2(z)=f_2(z)-\frac{(f_2,P_0)}{(P_0,P_0)}P_0(z)-\frac{(f_2,P_1)}{(P_1,P_1)}P_1(z)$$

where now we have $(f_2,P_1)=(z^2,z)=\int_{-1}^{1}z^3\,dz=0$, and

$$(f_2,P_0)=(z^2,1)=\int_{-1}^{1}z^2\,dz=\frac{2}{3}\,,\quad (P_0,P_0)=(1,1)=\int_{-1}^{1}dz=2$$

Therefore,

$$P_2(z)=z^2-\frac{2/3}{2}=z^2-\frac{1}{3}$$

Then, normalize it such that $P_2(1)=1$, and so on. For our discussion, we are going to renormalize the Legendre polynomials to unit norm. Because of (G.9), this amounts to multiplying the standard $P_n(z)$ by the factor $\sqrt{(2n+1)/2}$. Thus, we re-define:

$$P_n(z)=\sqrt{\frac{2n+1}{2}}\,\frac{1}{2^n n!}\frac{d^n}{dz^n}\big[(z^2-1)^n\big]\,,\quad n=0,1,2,\dots \tag{G.11}$$

Thus, (G.9) becomes $(P_n,P_m)=\delta_{nm}$. In particular, we note that now

$$P_0(z)=\frac{1}{\sqrt{2}} \tag{G.12}$$

By introducing the same scaling factors into each term of the recurrence (G.10), we find that the renormalized $P_n(z)$ satisfy:

$$zP_n(z)=\alpha_n P_{n-1}(z)+\alpha_{n+1}P_{n+1}(z)\,,\qquad \alpha_n=\frac{n}{\sqrt{4n^2-1}} \tag{G.13}$$

This relationship can be assumed to be valid also at $n=0$, provided we define $P_{-1}(z)=0$. For each order n, the Gram-Schmidt procedure replaces the non-orthogonal monomial basis by the orthonormalized Legendre basis:

$$\{1,\,z,\,z^2,\,\dots,\,z^n\}\ \Leftrightarrow\ \{P_0(z),\,P_1(z),\,P_2(z),\,\dots,\,P_n(z)\}$$

Thus, any polynomial $Q(z)$ of degree n can be expanded uniquely in either basis:

$$Q(z)=\sum_{k=0}^{n}q_k z^k=\sum_{k=0}^{n}c_k P_k(z)$$

with the expansion coefficients calculated from $c_k=(Q,P_k)$. This also implies that if $Q(z)$ has order $n-1$ then, it will be orthogonal to $P_n(z)$.

Next, we turn to the proof of the basic Gauss-Legendre result (G.5). Given a polynomial $P(z)$ of order $2N-1$, we can expand it uniquely in the form:

$$P(z)=P_N(z)Q(z)+R(z) \tag{G.14}$$

where $Q(z)$ and $R(z)$ are the quotient and remainder of the division by the Legendre polynomial $P_N(z)$, and both will have order $N-1$. Then, the integral of $P(z)$ can be written in inner-product notation as follows:

$$\int_{-1}^{1}P(z)\,dz=(P,1)=(P_NQ+R,1)=(P_NQ,1)+(R,1)=(Q,P_N)+(R,1)$$

But $(Q,P_N)=0$ because $Q(z)$ has order $N-1$ and $P_N(z)$ is orthogonal to all such polynomials. Thus, the integral of $P(z)$ can be expressed only in terms of the integral of the remainder polynomial $R(z)$, which has order $N-1$:

$$\int_{-1}^{1}P(z)\,dz=(P,1)=(R,1)=\int_{-1}^{1}R(z)\,dz \tag{G.15}$$

The right-hand side of the integration rule (G.5) can also be expressed in terms of $R(z)$:

$$\sum_{i=1}^{N}w_iP(z_i)=\sum_{i=1}^{N}w_iP_N(z_i)Q(z_i)+\sum_{i=1}^{N}w_iR(z_i) \tag{G.16}$$

and, because we assumed that $P_N(z_i)=0$,

$$\sum_{i=1}^{N}w_iP(z_i)=\sum_{i=1}^{N}w_iR(z_i) \tag{G.17}$$

Thus, combining (G.15) and (G.17), we obtain the following condition, which is equivalent to Eq. (G.5),

$$\int_{-1}^{1}R(z)\,dz=\sum_{i=1}^{N}w_iR(z_i) \tag{G.18}$$

Because $R(z)$ is an arbitrary polynomial of degree $N-1$, and has only N coefficients, this condition can be satisfied with a common set of N weights w_i for all such $R(z)$. If we had not assumed initially that the z_i were the zeros of $P_N(z)$, and took them to be an arbitrary set of N distinct points in $[-1,1]$, then (G.18) would read as

$$\int_{-1}^{1} R(z)\,dz = \sum_{i=1}^{N} w_i P_N(z_i) Q(z_i) + \sum_{i=1}^{N} w_i R(z_i)$$

In order for this to be satisfied for all $R(z)$ and all $Q(z)$, then (G.18) must still be satisfied by setting $Q(z)=0$, which fixes the weights w_i. Therefore, the first term in the right-hand side must be zero for all polynomials $Q(z)$ of degree $N-1$, and one can show that his implies that $P_N(z_i)=0$, that is, the z_i must be the zeros of $P_N(z)$.

Condition (G.18) can be used to determine the weights by expanding $R(z)$ into either the monomial basis or the Legendre basis, that is, because $R(z)$ has degree $N-1$:

$$R(z) = \sum_{k=0}^{N-1} r_k z^k = \sum_{k=0}^{N-1} c_k P_k(z) \tag{G.19}$$

Inserting, for example, the monomial basis into (G.18) and matching the coefficients of r_k on either side, we obtain the system of N equations for the weights:

$$\sum_{i=1}^{N} z_i^k w_i = \int_{-1}^{1} z^k\,dz = \frac{1+(-1)^k}{k+1}\,, \quad k = 0,1,\dots,N-1 \tag{G.20}$$

Defining the matrix $F_{ki} = z_i^k$ and the vector $u_k = \big[1+(-1)^k\big]/(k+1)$, we may write (G.20) in the compact matrix form:

$$F\mathbf{w} = \mathbf{u} \quad \Rightarrow \quad \mathbf{w} = F^{-1}\mathbf{u} \tag{G.21}$$

Alternatively, we may use the Legendre basis, which is more elegant. The left hand side of (G.18) will receive contribution only from the $k=0$ term because P_0 is orthogonal to all the succeeding P_k. Indeed, using the definition (G.12), we have:

$$\int_{-1}^{1} R(z)\,dz = (R,1) = \sqrt{2}\,(R,P_0) = \sqrt{2}\sum_{k=0}^{N-1} c_k (P_K,P_0) = \sqrt{2}\sum_{k=0}^{N-1} c_k \delta_{k0} = \sqrt{2}\,c_0$$

The right-hand side of (G.18) may be written as follows. Defining the $N\times N$ matrix $P_{ki} = P_k(z_i)$, $i = 1,2\dots,N$, and $k = 0,1,\dots,N-1$, and the row vector $\mathbf{c}^T = [c_0,c_1,\dots,c_{N-1}]$ of expansion coefficients, we have,

$$\sum_{i=1}^{N} w_i R(z_i) = \sum_{k=0}^{N-1}\sum_{i=1}^{N} c_k P_k(z_i) w_i = \mathbf{c}^T P\mathbf{w}$$

Thus, (G.18) now reads, where $\mathbf{u}_0 = [1,0,0,\dots,0]^T$:

$$\mathbf{c}^T P\mathbf{w} = \sqrt{2}\,c_0 = \sqrt{2}\,\mathbf{c}^T\mathbf{u}_0$$

Because the vector $\mathbf{c}$ is arbitrary, we must have the condition:

$$P\mathbf{w} = \sqrt{2}\,\mathbf{u}_0 \quad \Rightarrow \quad \mathbf{w} = \sqrt{2}\,P^{-1}\mathbf{u}_0 \tag{G.22}$$

The matrix P has some rather interesting properties. First, it has mutually orthogonal columns. Second, these columns are the eigenvectors of a Hermitian tridiagonal matrix whose eigenvalues are the zeros z_i. Thus, the problem of finding both z_i and w_i is reduced to an eigenvalue problem.

These eigenvalue properties follow from the recursion (G.13) of the normalized Legendre polynomials. For $n = 0,1,2,3$, the recursion reads explicitly:

$$\begin{aligned}
zP_0(z) &= \alpha_1 P_1(z)\\
zP_1(z) &= \alpha_1 P_0(z) + \alpha_2 P_2(z)\\
zP_2(z) &= \alpha_2 P_1(z) + \alpha_3 P_3(z)\\
zP_3(z) &= \alpha_3 P_2(z) + \alpha_4 P_4(z)
\end{aligned}$$

which can be written in matrix form:

$$z\begin{bmatrix} P_0(z)\\ P_1(z)\\ P_2(z)\\ P_3(z)\end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & 0 & 0\\ \alpha_1 & 0 & \alpha_2 & 0\\ 0 & \alpha_2 & 0 & \alpha_3\\ 0 & 0 & \alpha_3 & 0\end{bmatrix}\begin{bmatrix} P_0(z)\\ P_1(z)\\ P_2(z)\\ P_3(z)\end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 0\\ \alpha_4 P_4(z)\end{bmatrix}$$

and more generally,

$$z\begin{bmatrix} P_0(z)\\ P_1(z)\\ P_2(z)\\ \vdots\\ P_{N-2}(z)\\ P_{N-1}(z)\end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & \cdots & 0\\ \alpha_1 & 0 & \alpha_2 & 0 & \cdots & 0\\ 0 & \alpha_2 & 0 & \alpha_3 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & \alpha_{N-2} & 0 & \alpha_{N-1}\\ 0 & \cdots & 0 & 0 & \alpha_{N-1} & 0\end{bmatrix}\begin{bmatrix} P_0(z)\\ P_1(z)\\ P_2(z)\\ \vdots\\ P_{N-2}(z)\\ P_{N-1}(z)\end{bmatrix} + \begin{bmatrix} 0\\ 0\\ 0\\ \vdots\\ 0\\ \alpha_N P_N(z)\end{bmatrix}$$

Now, if z is replaced by the ith zero z_i of $P_N(z)$, the last column will vanish and we obtain the eigenvalue equation:

$$\begin{bmatrix} 0 & \alpha_1 & 0 & 0 & \cdots & 0\\ \alpha_1 & 0 & \alpha_2 & 0 & \cdots & 0\\ 0 & \alpha_2 & 0 & \alpha_3 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & \alpha_{N-2} & 0 & \alpha_{N-1}\\ 0 & \cdots & 0 & 0 & \alpha_{N-1} & 0\end{bmatrix}\begin{bmatrix} P_0(z_i)\\ P_1(z_i)\\ P_2(z_i)\\ \vdots\\ P_{N-2}(z_i)\\ P_{N-1}(z_i)\end{bmatrix} = z_i\begin{bmatrix} P_0(z_i)\\ P_1(z_i)\\ P_2(z_i)\\ \vdots\\ P_{N-2}(z_i)\\ P_{N-1}(z_i)\end{bmatrix} \tag{G.23}$$

Denoting the above tridiagonal matrix by A and the column of $P_k(z_i)$'s by $\mathbf{p}_i$, we may write compactly:

$$A\mathbf{p}_i = z_i\mathbf{p}_i\,, \quad i = 1,2,\dots,N \tag{G.24}$$

Thus, the eigenvalues of A are the zeros z_i and the corresponding eigenvectors are the columns $\mathbf{p}_i$ of the matrix P that we introduced in (G.22). Because the zeros z_i are distinct and A is a Hermitian matrix, its eigenvectors will be mutually orthogonal:

$$\mathbf{p}_i^T \mathbf{p}_j = d_i^2 \delta_{ij} \tag{G.25}$$

where $d_i = \|\mathbf{p}_i\|$ are the norms of the vectors $\mathbf{p}_i$. It follows that the orthonormalized eigenvectors of A will be $\boldsymbol{v}_i = \mathbf{p}_i/d_i$, and the orthogonal matrix of eigenvectors having the $\boldsymbol{v}_i$ as columns will be $V = [\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_N]$, or, expressed in terms of the matrix P and the diagonal matrix $D = \text{diag}\{d_1, d_2, \dots, d_N\}$:

$$V = PD^{-1} \tag{G.26}$$

Replacing P in (G.22) by $P = VD$ and using the orthogonality $V^T V = I$ of the eigenvector matrix, or $V^{-1} = V^T$, we obtain the solution:

$$\mathbf{w} = \sqrt{2}\, D^{-1} V^T \mathbf{u}_0 \quad \Rightarrow \quad w_i = \sqrt{2}\, d_i^{-1} (\boldsymbol{v}_i^T \mathbf{u}_0) \tag{G.27}$$

The matrix D can itself be expressed in terms of V by noting that the top entry of $\mathbf{p}_i$ is $P_0(z_i) = 1/\sqrt{2}$, and therefore, it follows from $\boldsymbol{v}_i = \mathbf{p}_i/d_i$ that the top entry of $\boldsymbol{v}_i$ will be $\boldsymbol{v}_i^T \mathbf{u}_0 = 1/(\sqrt{2} d_i)$, or, $d_i^{-1} = \sqrt{2}(\boldsymbol{v}_i^T \mathbf{u}_0)$. It finally follows from Eq. (G.27) that

$$w_i = d_i^{-2} = 2(\boldsymbol{v}_i^T \mathbf{u}_0)^2 \tag{G.28}$$

In MATLAB language, $\boldsymbol{v}_i^T \mathbf{u}_0 = V(1, i)$, that is, the first row of V. Because the eigenvectors of the Hermitian matrix A are real-valued and unique up to a sign, Eq. (G.28) allows the unique determination of the weights from the eigenvector matrix V.

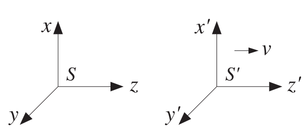
The above discussion leads to two possible implementations of the MATLAB function `quadr`. In the first, we obtain the coefficients of the Legendre polynomial $P_N(z)$, find its zeros using the built-in function `root`, and then solve the linear equation (G.21) for the weights. The second approach, implemented by the function `quadr2` and the related function `quadrs2`, determines z_i, w_i from the eigenvalue problem of the matrix A.

H. Lorentz Transformations

According to Einstein's special theory of relativity [435], Lorentz transformations describe the transformation between the space-time coordinates of two coordinate systems moving relative to each other at constant velocity. Maxwell's equations remain invariant under Lorentz transformations. This is demonstrated below.

Let the two coordinate frames be S and S'. By convention, we may think of S as the "fixed" laboratory frame with respect to which the frame S' is moving at a constant velocity $\boldsymbol{v}$. For example, if $\boldsymbol{v}$ is in the z-direction, the space-time coordinates $\{t, x, y, z\}$ of S are related to the coordinates $\{t', x', y', z'\}$ of S' by the Lorentz transformation:

$$\boxed{\begin{aligned} t' &= \gamma\Big(t - \frac{v}{c^2} z\Big) \\ z' &= \gamma(z - vt) \\ x' &= x \\ y' &= y \end{aligned}}\,, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$



where c is the speed of light in vacuum. Defining the scaled quantities $\tau = ct$ and $\beta = v/c$, the above transformation and its inverse, obtained by replacing β by $-\beta$, may be written as follows:

$$\boxed{\begin{aligned} \tau' &= \gamma(\tau - \beta z) \\ z' &= \gamma(z - \beta\tau) \\ x' &= x \\ y' &= y \end{aligned}} \quad \Leftrightarrow \quad \boxed{\begin{aligned} \tau &= \gamma(\tau' + \beta z') \\ z &= \gamma(z' + \beta\tau') \\ x &= x' \\ y &= y' \end{aligned}} \tag{H.1}$$

These transformations are also referred to as *Lorentz boosts* to indicate the fact that one frame is boosted to move relative to the other. Interchanging the roles of z and x, or z and y, one obtains the Lorentz transformations for motion along the x or y directions, respectively. Eqs. (H.1) may be expressed more compactly in matrix form:

$$\boxed{\mathrm{x}' = L\mathrm{x}}\,, \quad \text{where} \quad \mathrm{x} = \begin{bmatrix} \tau \\ x \\ y \\ z \end{bmatrix}, \quad \mathrm{x}' = \begin{bmatrix} \tau' \\ x' \\ y' \\ z' \end{bmatrix}, \quad L = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \tag{H.2}$$

Such transformations leave the quadratic form $(c^2t^2 - x^2 - y^2 - z^2)$ invariant, that is,

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{H.3}$$

Introducing the diagonal metric matrix $G = \text{diag}(1, -1, -1, -1)$, we may write the quadratic form as follows, where x^T denotes the transposed vector, that is, the row vector $\mathrm{x}^T = [\tau, x, y, z]$:

$$\mathrm{x}^T G \mathrm{x} = \tau^2 - x^2 - y^2 - z^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{H.4}$$

More generally, a Lorentz transformation is defined as any linear transformation $\mathrm{x}' = L\mathrm{x}$ that leaves the quadratic form $\mathrm{x}^T G \mathrm{x}$ invariant. The invariance condition requires that: $\mathrm{x}'^T G \mathrm{x}' = \mathrm{x}^T L^T G L \mathrm{x} = \mathrm{x}^T G \mathrm{x}$, or

$$\boxed{L^T G L = G} \tag{H.5}$$

In addition to the Lorentz boosts of Eq. (H.1), the more general transformations satisfying (H.5) include rotations of the three spatial coordinates, as well as time or space reflections. For example, a rotation has the form:

$$L = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right]$$

where R is a 3×3 orthogonal rotation matrix, that is, $R^T R = I$, where I is the 3×3 identity matrix. The most general Lorentz boost corresponding to arbitrary velocity $\boldsymbol{v} = [v_x, v_y, v_z]^T$ is given by:

$$L = \left[\begin{array}{c|c} \gamma & -\gamma\boldsymbol{\beta}^T \\ \hline -\gamma\boldsymbol{\beta} & I + \dfrac{\gamma^2}{\gamma + 1}\boldsymbol{\beta}\boldsymbol{\beta}^T \end{array}\right], \quad \text{where} \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^T\boldsymbol{\beta}}} \tag{H.6}$$

When $\boldsymbol{v} = [0,0,v]^T$, or $\boldsymbol{\beta} = [0,0,\beta]^T$, Eq. (H.6) reduces to (H.1). Defining $\beta = |\boldsymbol{\beta}| = \sqrt{\boldsymbol{\beta}^T\boldsymbol{\beta}}$ and the unit vector $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}/\beta$, and using the relationship $\gamma^2\beta^2 = \gamma^2 - 1$, it can be verified that the spatial part of the matrix L can be written in the form:

$$I + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\boldsymbol{\beta}^T = I + (\gamma-1)\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \tag{H.7}$$

The set of matrices L satisfying Eq. (H.5) forms a group called the *Lorentz group*. In particular, the z-directed boosts of Eq. (H.2) form a commutative subgroup. Denoting these boosts by $L(\beta)$, the application of two successive boosts by velocity factors $\beta_1 = v_1/c$ and $\beta_2 = v_2/c$ leads to the combined boost $L(\beta) = L(\beta_1)L(\beta_2)$, where:

$$\boxed{\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}} \quad\Leftrightarrow\quad \boxed{v = \frac{v_1+v_2}{1+v_1v_2/c^2}} \tag{H.8}$$

with $\beta = v/c$. Eq. (H.8) is Einstein's relativistic velocity addition theorem. The same group property implies also that $L^{-1}(\beta) = L(-\beta)$. The proof of Eq. (H.8) follows from the following condition, where $\gamma_1 = 1/\sqrt{1-\beta_1^2}$ and $\gamma_2 = 1/\sqrt{1-\beta_2^2}$:

$$\begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 & 0 & -\gamma_1\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1\beta_1 & 0 & 0 & \gamma_1 \end{bmatrix}\begin{bmatrix} \gamma_2 & 0 & 0 & -\gamma_2\beta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_2\beta_2 & 0 & 0 & \gamma_2 \end{bmatrix}$$

A *four-vector* is a four-dimensional vector that transforms like the vector x under Lorentz transformations, that is, its components with respect to the two moving frames S and S' are related by:

$$\boxed{a' = La}, \quad \text{where } a = \begin{bmatrix} a_0 \\ a_x \\ a_y \\ a_z \end{bmatrix}, \quad a' = \begin{bmatrix} a_0' \\ a_x' \\ a_y' \\ a_z' \end{bmatrix} \tag{H.9}$$

For example, under the z-directed boost of Eq. (H.1), the four-vector a will transform as:

$$\boxed{\begin{aligned} a_0' &= \gamma(a_0 - \beta a_z) \\ a_z' &= \gamma(a_z - \beta a_0) \\ a_x' &= a_x \\ a_y' &= a_y \end{aligned}} \quad\Leftrightarrow\quad \boxed{\begin{aligned} a_0 &= \gamma(a_0' + \beta a_z') \\ a_z &= \gamma(a_z' + \beta a_0') \\ a_x &= a_x' \\ a_y &= a_y' \end{aligned}} \tag{H.10}$$

Four-vectors transforming according to Eq. (H.9) are referred to as *contravariant*. Under the general Lorentz boost of Eq. (H.6), the spatial components of a that are *transverse* to the direction of the velocity vector $\boldsymbol{v}$ remain *unchanged*, whereas the *parallel* component transforms as in Eq. (H.10), that is, the most general Lorentz boost transformation for a four-vector takes the form:

$$\boxed{\begin{aligned} a_0' &= \gamma(a_0 - \beta a_\parallel) \\ a_\parallel' &= \gamma(a_\parallel - \beta a_0) \\ \boldsymbol{a}_\perp' &= \boldsymbol{a}_\perp \end{aligned}} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = |\boldsymbol{\beta}|, \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} \tag{H.11}$$

where $a_\parallel = \hat{\boldsymbol{\beta}}^T\boldsymbol{a}$ and $\boldsymbol{a} = [a_x, a_y, a_z]^T$ is the spatial part of a. Then,

$$\boldsymbol{a}_\parallel = \hat{\boldsymbol{\beta}}a_\parallel = \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}^T\boldsymbol{a}) \quad\text{and}\quad \boldsymbol{a}_\perp = \boldsymbol{a} - \boldsymbol{a}_\parallel = \boldsymbol{a} - \hat{\boldsymbol{\beta}}a_\parallel$$

Setting $\boldsymbol{\beta} = \beta\hat{\boldsymbol{\beta}}$ and using Eq. (H.7), the Lorentz transformation (H.6) gives:

$$\begin{bmatrix} a_0' \\ \boldsymbol{a}' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta\hat{\boldsymbol{\beta}}^T \\ -\gamma\beta\hat{\boldsymbol{\beta}} & I + (\gamma-1)\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}^T \end{bmatrix}\begin{bmatrix} a_0 \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} \gamma(a_0 - \beta a_\parallel) \\ \boldsymbol{a} - \hat{\boldsymbol{\beta}}a_\parallel + \hat{\boldsymbol{\beta}}\gamma(a_\parallel - \beta a_0) \end{bmatrix}$$

from which Eq. (H.11) follows.

For any two four-vectors a, b, the quadratic form a^TGb remains invariant under Lorentz transformations, that is, $a'^TGb' = a^TGb$, or,

$$a_0'b_0' - \boldsymbol{a}'\cdot\boldsymbol{b}' = a_0b_0 - \boldsymbol{a}\cdot\boldsymbol{b}, \quad \text{where} \quad a = \begin{bmatrix} a_0 \\ \boldsymbol{a} \end{bmatrix}, \quad b = \begin{bmatrix} b_0 \\ \boldsymbol{b} \end{bmatrix} \tag{H.12}$$

Some examples of four-vectors are given in the following table:

four-vector	a_0	a_x	a_y	a_z
time and space	ct	x	y	z
frequency and wavenumber	ω/c	k_x	k_y	k_z
energy and momentum	E/c	p_x	p_y	p_z
charge and current densities	$c\rho$	J_x	J_y	J_z
scalar and vector potentials	φ	cA_x	cA_y	cA_z

(H.13)

For example, under the z-directed boost of Eq. (H.1), the frequency-wavenumber transformation will be as follows:

$$\boxed{\begin{aligned} \omega' &= \gamma(\omega - \beta ck_z) \\ k_z' &= \gamma(k_z - \frac{\beta}{c}\omega) \\ k_x' &= k_x \\ k_y' &= k_y \end{aligned}} \quad\Leftrightarrow\quad \boxed{\begin{aligned} \omega &= \gamma(\omega' + \beta ck_z') \\ k_z &= \gamma(k_z' + \frac{\beta}{c}\omega') \\ k_x &= k_x' \\ k_y &= k_y' \end{aligned}}, \quad \beta c = v, \quad \frac{\beta}{c} = \frac{v}{c^2} \tag{H.14}$$

where we rewrote the first equations in terms of ω instead of ω/c. The change in frequency due to motion is the basis of the Doppler effect. The invariance property (H.12) applied to the space-time and frequency-wavenumber four-vectors reads:

$$\omega't' - \boldsymbol{k}'\cdot\mathbf{r}' = \omega t - \boldsymbol{k}\cdot\mathbf{r} \tag{H.15}$$

This implies that a uniform plane wave remains a uniform plane wave in all reference frames moving at a constant velocity relative to each other. Similarly, the charge and current densities transform as follows:

$$\boxed{\begin{aligned} c\rho' &= \gamma(c\rho - \beta J_z) \\ J_z' &= \gamma(J_z - \beta c\rho) \\ J_x' &= J_x \\ J_y' &= J_y \end{aligned}} \quad\Leftrightarrow\quad \boxed{\begin{aligned} c\rho &= \gamma(c\rho' + \beta J_z') \\ J_z &= \gamma(J_z' + \beta c\rho') \\ J_x &= J_x' \\ J_y &= J_y' \end{aligned}} \tag{H.16}$$

Because Eq. (H.5) implies that $L^{-T} = GLG$, we are led to define four-vectors that transform according to L^{-T}. Such four-vectors are referred to as being *covariant*. Given any contravariant 4-vector a, we define its covariant version by $\bar{a} = Ga$. This operation simply reverses the sign of the spatial part of a:

$$\bar{a} = Ga = \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} a_0 \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} a_0 \\ -\boldsymbol{a} \end{bmatrix} \tag{H.17}$$

The vector $\bar{a}$ transforms as follows:

$$\bar{a}' = Ga' = GLa = (GLG)(Ga) = L^{-T}\bar{a} \tag{H.18}$$

where we used the property that $G^2 = I_4$, the 4×4 identity matrix. The most important covariant vector is the four-dimensional gradient:

$$\partial_{\mathsf{x}} = \begin{bmatrix} \partial_\tau \\ \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} = \begin{bmatrix} \partial_\tau \\ \boldsymbol{\nabla} \end{bmatrix} \tag{H.19}$$

Because $\mathsf{x}' = L\mathsf{x}$, it follows that $\partial_{\mathsf{x}'} = L^{-T}\partial_{\mathsf{x}}$. Indeed, we have component-wise:

$$\frac{\partial}{\partial \mathsf{x}_i} = \sum_j \frac{\partial \mathsf{x}'_j}{\partial \mathsf{x}_i}\frac{\partial}{\partial \mathsf{x}'_j} = \sum_j L_{ji}\frac{\partial}{\partial \mathsf{x}'_j} \quad \Rightarrow \quad \partial_{\mathsf{x}} = L^T\partial_{\mathsf{x}'} \quad \Rightarrow \quad \partial_{\mathsf{x}'} = L^{-T}\partial_{\mathsf{x}}$$

For the z-directed boost of Eq. (H.1), we have $L^{-T} = L^{-1}$, which gives:

$$\boxed{\begin{aligned} \partial_{\tau'} &= \gamma(\partial_\tau + \beta\partial_z) \\ \partial_{z'} &= \gamma(\partial_z + \beta\partial_\tau) \\ \partial_{x'} &= \partial_x \\ \partial_{y'} &= \partial_y \end{aligned}} \quad \Leftrightarrow \quad \boxed{\begin{aligned} \partial_\tau &= \gamma(\partial_{\tau'} - \beta\partial_{z'}) \\ \partial_z &= \gamma(\partial_{z'} - \beta\partial_{\tau'}) \\ \partial_x &= \partial_{x'} \\ \partial_y &= \partial_{y'} \end{aligned}} \tag{H.20}$$

The four-dimensional divergence of a four-vector is a Lorentz scalar. For example, denoting the current density four-vector by $J = [c\rho, J_x, J_y, J_z]^T$, the charge conservation law involves the four-dimensional divergence:

$$\partial_t\rho + \boldsymbol{\nabla}\cdot\boldsymbol{J} = [\partial_\tau, \partial_x, \partial_y, \partial_z]\begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix} = \partial_{\mathsf{x}}^T J \tag{H.21}$$

Under a Lorentz transformation, this remains invariant, and therefore, if it is zero in one frame it will remain zero in all frames. Using $\partial_{\mathsf{x}}^T = \partial_{\mathsf{x}'}^T L$, we have:

$$\partial_t\rho + \boldsymbol{\nabla}\cdot\boldsymbol{J} = \partial_{\mathsf{x}}^T J = \partial_{\mathsf{x}'}^T LJ = \partial_{\mathsf{x}'} J' = \partial_{t'}\rho' + \boldsymbol{\nabla}'\cdot\boldsymbol{J}' \tag{H.22}$$

Although many quantities in electromagnetism transform like four-vectors, such as the space-time or the frequency-wavenumber vectors, the actual electromagnetic fields do not. Rather, they transform like six-vectors or rank-2 antisymmetric tensors.

A rank-2 tensor is represented by a 4×4 matrix, say F. Its Lorentz transformation properties are the same as the transformation of the product of a column and a row four-vector, that is, F transforms like the quantity ab^T, where a, b are column four-vectors. This product transforms like $a'b'^T = L(ab^T)L^T$. Thus, a general second-rank tensor transforms as follows:

$$\boxed{F' = LFL^T} \tag{H.23}$$

An *antisymmetric* rank-2 tensor F defines, and is completely defined by, two three-dimensional vectors, say $\boldsymbol{a} = [a_x, a_y, a_z]^T$ and $\boldsymbol{b} = [b_x, b_y, b_z]^T$. Its matrix form is:

$$F = \begin{bmatrix} 0 & -a_x & -a_y & -a_z \\ a_x & 0 & -b_z & b_y \\ a_y & b_z & 0 & -b_x \\ a_z & -b_y & b_x & 0 \end{bmatrix} \tag{H.24}$$

Given the tensor F, one may define its *covariant* version through $\bar{F} = GFG$, and its *dual*, denoted by $\tilde{F}$ and obtained by the replacements $\boldsymbol{a} \to \boldsymbol{b}$ and $\boldsymbol{b} \to -\boldsymbol{a}$, that is,

$$\bar{F} = \begin{bmatrix} 0 & a_x & a_y & a_z \\ -a_x & 0 & -b_z & b_y \\ -a_y & b_z & 0 & -b_x \\ -a_z & -b_y & b_x & 0 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 0 & -b_x & -b_y & -b_z \\ b_x & 0 & a_z & -a_y \\ b_y & -a_z & 0 & a_x \\ b_z & a_y & -a_x & 0 \end{bmatrix} \tag{H.25}$$

Thus, $\bar{F}$ corresponds to the pair $(-\boldsymbol{a}, \boldsymbol{b})$, and $\tilde{F}$ to $(\boldsymbol{b}, -\boldsymbol{a})$. Their Lorentz transformation properties are:

$$\bar{F}' = L^{-T}\bar{F}L^{-1}, \quad \tilde{F}' = L\tilde{F}L^T \tag{H.26}$$

Thus, the dual $\tilde{F}$ transforms like F itself. For the z-directed boost of Eq. (H.1), it follows from (H.23) that the two vectors $\boldsymbol{a}, \boldsymbol{b}$ transform as follows:

$$\boxed{\begin{aligned} a'_x &= \gamma(a_x - \beta b_y) & \quad b'_x &= \gamma(b_x + \beta a_y) \\ a'_y &= \gamma(a_y + \beta b_x) & \quad b'_y &= \gamma(b_y - \beta a_x) \\ a'_z &= a_z & \quad b'_z &= b_z \end{aligned}} \tag{H.27}$$

These are obtained by equating the expressions:

$$\begin{bmatrix} 0 & -a'_x & -a'_y & -a'_z \\ a'_x & 0 & -b'_z & b'_y \\ a'_y & b'_z & 0 & -b'_x \\ a'_z & -b'_y & b'_x & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 & -a_x & -a_y & -a_z \\ a_x & 0 & -b_z & b_y \\ a_y & b_z & 0 & -b_x \\ a_z & -b_y & b_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix}$$

More generally, under the boost transformation (H.6), it can be verified that the components of $\boldsymbol{a}, \boldsymbol{b}$ parallel and perpendicular to $\boldsymbol{v}$ transform as follows:

$$\boxed{\begin{aligned} \boldsymbol{a}'_\perp &= \gamma(\boldsymbol{a}_\perp + \boldsymbol{\beta}\times\boldsymbol{b}_\perp) \\ \boldsymbol{b}'_\perp &= \gamma(\boldsymbol{b}_\perp - \boldsymbol{\beta}\times\boldsymbol{a}_\perp) \\ \boldsymbol{a}'_\parallel &= \boldsymbol{a}_\parallel \\ \boldsymbol{b}'_\parallel &= \boldsymbol{b}_\parallel \end{aligned}} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = |\boldsymbol{\beta}|, \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c} \tag{H.28}$$

Thus, in contrast to Eq. (H.11) for a four-vector, the parallel components remain unchanged while the transverse components change. A pair of three-dimensional vectors $(\boldsymbol{a}, \boldsymbol{b})$ transforming like Eq. (H.28) is referred to as a *six-vector*.

It is evident also that Eqs. (H.28) remain invariant under the duality transformation $\boldsymbol{a} \to \boldsymbol{b}$ and $\boldsymbol{b} \to -\boldsymbol{a}$, which justifies Eq. (H.26). Some examples of $(\boldsymbol{a}, \boldsymbol{b})$ six-vector pairs defining an antisymmetric rank-2 tensor are as follows:

$$\begin{array}{cc} \boldsymbol{a} & \boldsymbol{b} \\ \hline \boldsymbol{E} & c\boldsymbol{B} \\ c\boldsymbol{D} & \boldsymbol{H} \\ c\boldsymbol{P} & -\boldsymbol{M} \end{array} \tag{H.29}$$

where $\boldsymbol{P}, \boldsymbol{M}$ are the polarization and magnetization densities defined through the relationships $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$ and $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$. Thus, the $(\boldsymbol{E}, \boldsymbol{B})$ and $(\boldsymbol{D}, \boldsymbol{H})$ fields have the following Lorentz transformation properties:

$$\boxed{\begin{aligned} \boldsymbol{E}'_\perp &= \gamma(\boldsymbol{E}_\perp + c\boldsymbol{\beta}\times\boldsymbol{B}_\perp) \\ \boldsymbol{B}'_\perp &= \gamma(\boldsymbol{B}_\perp - \frac{1}{c}\boldsymbol{\beta}\times\boldsymbol{E}_\perp) \\ \boldsymbol{E}'_\parallel &= \boldsymbol{E}_\parallel \\ \boldsymbol{B}'_\parallel &= \boldsymbol{B}_\parallel \end{aligned}} \qquad \boxed{\begin{aligned} \boldsymbol{H}'_\perp &= \gamma(\boldsymbol{H}_\perp - c\boldsymbol{\beta}\times\boldsymbol{D}_\perp) \\ \boldsymbol{D}'_\perp &= \gamma(\boldsymbol{D}_\perp + \frac{1}{c}\boldsymbol{\beta}\times\boldsymbol{H}_\perp) \\ \boldsymbol{H}'_\parallel &= \boldsymbol{H}_\parallel \\ \boldsymbol{D}'_\parallel &= \boldsymbol{D}_\parallel \end{aligned}} \tag{H.30}$$

where we may replace $c\boldsymbol{\beta} = \boldsymbol{v}$ and $\boldsymbol{\beta}/c = \boldsymbol{v}/c^2$. Note that the two groups of equations transform into each other under the usual duality transformations: $\boldsymbol{E} \to \boldsymbol{H}$, $\boldsymbol{H} \to -\boldsymbol{E}$, $\boldsymbol{D} \to \boldsymbol{B}$, $\boldsymbol{B} \to -\boldsymbol{D}$. For the z-directed boost of Eq. (H.1), we have from Eq. (H.30):

$$\boxed{\begin{aligned} E'_x &= \gamma(E_x - c\beta B_y) \\ E'_y &= \gamma(E_y + c\beta B_x) \\ B'_x &= \gamma(B_x + \frac{1}{c}\beta E_y) \\ B'_y &= \gamma(B_y - \frac{1}{c}\beta E_x) \\ E'_z &= E_z \\ B'_z &= B_z \end{aligned}} \qquad \boxed{\begin{aligned} H'_x &= \gamma(H_x + c\beta D_y) \\ H'_y &= \gamma(H_y - c\beta D_x) \\ D'_x &= \gamma(D_x - \frac{1}{c}\beta H_y) \\ D'_y &= \gamma(D_y + \frac{1}{c}\beta H_x) \\ H'_z &= H_z \\ D'_z &= D_z \end{aligned}} \tag{H.31}$$

Associated with a six-vector $(\boldsymbol{a}, \boldsymbol{b})$, there are two scalar invariants: the quantities $(\boldsymbol{a}\cdot\boldsymbol{b})$ and $(\boldsymbol{a}\cdot\boldsymbol{a} - \boldsymbol{b}\cdot\boldsymbol{b})$. Their invariance follows from Eq. (H.28). Thus, the scalars $(\boldsymbol{E}\cdot\boldsymbol{B})$, $(\boldsymbol{E}\cdot\boldsymbol{E} - c^2\boldsymbol{B}\cdot\boldsymbol{B})$, $(\boldsymbol{D}\cdot\boldsymbol{H})$, $(c^2\boldsymbol{D}\cdot\boldsymbol{D} - \boldsymbol{H}\cdot\boldsymbol{H})$ remain invariant under Lorentz transformations. In addition, it follows from (H.30) that the quantity $(\boldsymbol{E}\cdot\boldsymbol{D} - \boldsymbol{B}\cdot\boldsymbol{H})$ is invariant.

Given a six-vector $(\boldsymbol{a}, \boldsymbol{b})$ and its dual $(\boldsymbol{b}, -\boldsymbol{a})$, we may define the following four-dimensional "current" vectors that are dual to each other:

$$J = \begin{bmatrix} \nabla\cdot\boldsymbol{a} \\ \nabla\times\boldsymbol{b} - \partial_\tau\boldsymbol{a} \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} \nabla\cdot\boldsymbol{b} \\ -\nabla\times\boldsymbol{a} - \partial_\tau\boldsymbol{b} \end{bmatrix} \tag{H.32}$$

It can be shown that both J and $\tilde{J}$ transform as four-vectors under Lorentz transformations, that is, $J' = LJ$ and $\tilde{J}' = L\tilde{J}$, where $J', \tilde{J}'$ are defined with respect to the coordinates of the S' frame:

$$J' = \begin{bmatrix} \nabla'\cdot\boldsymbol{a}' \\ \nabla'\times\boldsymbol{b}' - \partial_{\tau'}\boldsymbol{a}' \end{bmatrix}, \quad \tilde{J}' = \begin{bmatrix} \nabla'\cdot\boldsymbol{b}' \\ -\nabla'\times\boldsymbol{a}' - \partial_{\tau'}\boldsymbol{b}' \end{bmatrix} \tag{H.33}$$

The calculation is straightforward but tedious. For example, for the z-directed boost (H.1), we may use Eqs. (H.20) and (H.27) and the identity $\gamma^2(1-\beta^2) = 1$ to show:

$$\begin{aligned} J'_x &= (\nabla'\times\boldsymbol{b}' - \partial_{\tau'}\boldsymbol{a}')_x = \partial_{y'}b'_z - \partial_{z'}b'_y - \partial_{\tau'}a'_x \\ &= \partial_y b_z - \gamma^2(\partial_z + \beta\partial_\tau)(b_y - \beta a_x) - \gamma^2(\partial_\tau + \beta\partial_z)(a_x - \beta b_y) \\ &= \partial_y b_z - \partial_z b_y - \partial_\tau a_x = (\nabla\times\boldsymbol{b} - \partial_\tau\boldsymbol{a})_x = J_x \end{aligned}$$

Similarly, we have:

$$\begin{aligned} J'_0 &= \nabla'\cdot\boldsymbol{a}' = \partial_{x'}a'_x + \partial_{y'}a'_y + \partial_{z'}a'_z \\ &= \gamma\partial_x(a_x - \beta b_y) + \gamma\partial_y(a_y + \beta b_x) + \gamma(\partial_z + \beta\partial_\tau)a_z \\ &= \gamma\big[(\partial_x a_x + \partial_y a_y + \partial_z a_z) - \beta(\partial_x b_y - \partial_y b_x - \partial_\tau a_z)\big] = \gamma(J_0 - \beta J_z) \end{aligned}$$

In this fashion, one can show that J and $\tilde{J}$ satisfy the Lorentz transformation equations (H.10) for a four-vector. To see the significance of this result, we rewrite Maxwell's equations, with magnetic charge and current densities $\rho_m, \boldsymbol{J}_m$ included, in the four-dimensional forms:

$$\begin{bmatrix} \nabla\cdot c\boldsymbol{D} \\ \nabla\times\boldsymbol{H} - \partial_\tau c\boldsymbol{D} \end{bmatrix} = \begin{bmatrix} c\rho \\ \boldsymbol{J} \end{bmatrix}, \quad \begin{bmatrix} \nabla\cdot c\boldsymbol{B} \\ -\nabla\times\boldsymbol{E} - \partial_\tau c\boldsymbol{B} \end{bmatrix} = \begin{bmatrix} c\rho_m \\ \boldsymbol{J}_m \end{bmatrix} \tag{H.34}$$

Thus, applying the above result to the six-vector $(c\boldsymbol{D}, \boldsymbol{H})$ and to the dual of $(\boldsymbol{E}, c\boldsymbol{B})$ and assuming that the electric and magnetic current densities transform like four-vectors, it follows that Maxwell's equations remain invariant under Lorentz transformations, that is, they retain their form in the moving system:

$$\begin{bmatrix} \nabla'\cdot c\boldsymbol{D}' \\ \nabla'\times\boldsymbol{H}' - \partial_{\tau'} c\boldsymbol{D}' \end{bmatrix} = \begin{bmatrix} c\rho' \\ \boldsymbol{J}' \end{bmatrix}, \quad \begin{bmatrix} \nabla'\cdot c\boldsymbol{B}' \\ -\nabla'\times\boldsymbol{E}' - \partial_{\tau'} c\boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} c\rho'_m \\ \boldsymbol{J}'_m \end{bmatrix} \tag{H.35}$$

The Lorentz transformation properties of the electromagnetic fields allow one to solve problems involving moving media, such as the Doppler effect, reflection and transmission from moving boundaries, and so on. The main technique for solving such problems is to transform to the frame (here, S') in which the boundary is at rest, solve the reflection problem in that frame, and transform the results back to the laboratory frame by using the inverse of Eq. (H.30).

This procedure was discussed by Einstein in his 1905 paper on special relativity in connection to the Doppler effect from a moving mirror. To quote [435]: "All problems in the optics of moving bodies can be solved by the method here employed. What is essential is that the electric and magnetic force of the light which is influenced by a moving body, be transformed into a system of co-ordinates at rest relatively to the body. By this means all problems in the optics of moving bodies will be reduced to a series of problems in the optics of stationary bodies."

I. MATLAB Functions

The MATLAB functions are grouped by category. They are available from the web page: `www.ece.rutgers.edu/~orfanidi/ewa`.

Multilayer Dielectric Structures

```
brewster   - calculates Brewster and critical angles
fresnel    - Fresnel reflection coefficients for isotropic or birefringent media

n2r        - refractive indices to reflection coefficients of M-layer structure
r2n        - reflection coefficients to refractive indices of M-layer structure

multidiel  - reflection response of isotropic or birefringent multilayer structures
multidiel1 - simplified version of multidiel for isotropic layers
multidiel2 - reflection response of lossy isotropic multilayer dielectric structures
omniband   - bandwidth of omnidirectional mirrors and Brewster polarizers
omniband2  - bandwidth of birefringent multilayer mirrors

snel       - calculates refraction angles from Snel's law for birefringent media
```

Quarter-Wavelength Transformers

```
bkwrec  - order-decreasing backward layer recursion - from a,b to r
frwrec  - order-increasing forward layer recursion - from r to A,B

chebtr  - Chebyshev broadband reflectionless quarter-wave transformer
chebtr2 - Chebyshev broadband reflectionless quarter-wave transformer
chebtr3 - Chebyshev broadband reflectionless quarter-wave transformer
```

Dielectric Waveguides

```
dguide - TE modes in dielectric slab waveguide
dslab  - solves for the TE-mode cutoff wavenumbers in a dielectric slab
```


Transmission Lines

```
g2z       - reflection coefficient to impedance transformation
z2g       - impedance to reflection coefficient transformation
lmin      - find locations of voltage minima and maxima

mstripa   - microstrip analysis (calculates Z,eff from w/h)
mstripr   - microstrip synthesis with refinement (calculates w/h from Z)
mstrips   - microstrip synthesis (calculates w/h from Z)

multiline - reflection response of multi-segment transmission line

swr       - standing wave ratio
tsection  - T-section equivalent of a length-l transmission line segment

gprop     - reflection coefficient propagation
vprop     - wave impedance propagation
zprop     - wave impedance propagation
```

Impedance Matching

```
qwt1     - quarter wavelength transformer with series segment
qwt2     - quarter wavelength transformer with 1/8-wavelength shunt stub
qwt3     - quarter wavelength transformer with shunt stub of adjustable length

dualband - two-section dual-band Chebyshev impedance transformer
dualbw   - two-section dual-band transformer bandwidths

stub1    - single-stub matching
stub2    - double-stub matching
stub3    - triple-stub matching

onesect  - one-section impedance transformer
twosect  - two-section impedance transformer

pi2t     - Pi to T transformation
t2pi     - Pi to T transformation
lmatch   - L-section reactive conjugate matching network
pmatch   - Pi-section reactive conjugate matching network
```

S-Parameters

```
gin      - input reflection coefficient in terms of S-parameters
gout     - output reflection coefficient in terms of S-parameters
nfcirc   - constant noise figure circle
nfig     - noise figure of two-port
sgain    - transducer, available, and operating power gains of two-port
sgcirc   - stability and gain circles
smat     - S-parameters to S-matrix
smatch   - simultaneous conjugate match of a two-port
smith    - draw basic Smith chart
smithcir - add stability and constant gain circles on Smith chart
sparam   - stability parameters of two-port
circint  - circle intersection on Gamma-plane
circtan  - point of tangency between the two circles
```

Linear Antenna Functions

```
dipdir    - dipole directivity
dmax      - computes directivity and beam solid angle of g(th) gain
dipole    - gain of center-fed linear dipole of length L
traveling - gain of traveling-wave antenna of length L
vee       - gain of traveling-wave vee antenna
rhombic   - gain of traveling-wave rhombic antenna

king      - King's 3-term sinusoidal approximation
kingeval  - evaluate King's 3-term sinusoidal current approximation
kingfit   - fits a sampled current to King's 2-term sinusoidal approximation
kingprime - converts King's 3-term coefficients from unprimed to primed form

hbasis    - basis functions for Hallen equation
hdelta    - solve Hallen's equation with delta-gap input
hfield    - solve Hallen's equation with arbitrary incident E-field
hmat      - Hallen impedance matrix with method of moments and point-matching
hwrap     - wraps a Toeplitz impedance matrix to half its size
kernel    - thin-wire kernel computation for Hallen equation
pfield    - solve Pocklington's equation with arbitrary incident E-field
pmat      - Pocklington impedance matrix with method of moments and point-matching

hcoupled  - solve Hallen's equation for 2D array of non-identical parallel dipoles
hcoupled2 - solve Hallen's equation for 2D array of identical parallel dipoles

gain2d    - normalized gain of 2D array of parallel dipoles with Hallen currents
gain2s    - normalized gain of 2D array of parallel dipoles with sinusoidal currents
imped     - mutual impedance between two parallel standing-wave dipoles
imped2    - mutual impedance between two parallel standing-wave dipoles
impedmat  - mutual impedance matrix of array of parallel dipole antennas
resonant  - calculates the length of a resonant dipole antenna
yagi      - simplified Yagi-Uda array design
```

Aperture Antenna Functions

```
diffint - generalized Fresnel diffraction integral
diffr   - knife-edge diffraction coefficient
dsinc   - the double-sinc function cos(pi*x)/(1-4*x^2)

fcs     - Fresnel integrals C(x) and S(x)
fcs2    - type-2 Fresnel integrals C2(x) and S2(x)

hband   - horn antenna 3-dB width
heff    - aperture efficiency of horn antenna
hgain   - horn antenna H-plane and E-plane gains
hopt    - optimum horn antenna design
hsigma  - optimum sigma parametes for horn antenna
```

Antenna Array Functions

```
gain1d    - normalized gain computation for 1D equally-spaced isotropic array
bwidth    - beamwidth mapping from psi-space to phi-space
binomial  - binomial array weights
dolph     - Dolph-Chebyshev array weights
dolph2    - Riblet-Pritchard version of Dolph-Chebyshev
dolph3    - DuHamel version of endfire Dolph-Chebyshev
```



```
multibeam - multibeam array design
prol      - prolate array design
prolmat   - prolate matrix
scan      - scan array with given scanning phase
sector    - sector beam array design
steer     - steer array towards given angle
taylor1n  - Taylor n-bar line source array design
taylor1p  - Taylor 1-parameter array design
taylorbw  - Taylor B-parameter and beamwidth
uniform   - uniform array weights
woodward  - Woodward-Lawson-Butler beams
ville     - Villeneuve array design

chebarray - Bresler's Chebyshev array design method (written by P. Simon)
```

Gain Plotting Functions

```
abp     - polar gain plot in absolute units
abz     - azimuthal gain plot in absolute units
ab2p    - polar gain plot in absolute units - 2*pi angle range
abz2    - azimuthal gain plot in absolute units - 2pi angle range

dbp     - polar gain plot in dB
dbz     - azimuthal gain plot in dB
dbp2    - polar gain plot in dB - 2*pi angle range
dbz2    - azimuthal gain plot in dB - 2pi angle range

abadd   - add gain in absolute units
abadd2  - add gain in absolute units - 2pi angle range
dbadd   - add gain in dB
dbadd2  - add gain in dB - 2pi angle range
addbwp  - add 3-dB angle beamwidth in polar plots
addbwz  - add 3-dB angle beamwidth in azimuthal plots
addcirc - add grid circle in polar or azimuthal plots
addline - add grid ray line in azimuthal or polar plots
addray  - add ray in azimuthal or polar plots
```

Miscellaneous Utility Functions

```
ab       - dB to absolute power units
db       - absolute power to dB units

c2p      - complex number to phasor form
p2c      - phasor form to complex number

d2r      - degrees to radians
r2d      - radians to degrees

dtft     - DTFT of a signal x at a frequency vector w
I0       - modified Bessel function of 1st kind and 0th order
ellipse  - polarization ellipse parameters
etac     - eta and c
wavenum  - calculate wavenumber and characteristic impedance
poly2    - specialized version of poly with increased accuracy

quadr    - Gauss-Legendre quadrature weights and evaluation points
quadrs   - quadrature weights and evaluation points on subintervals
```

```
quadr2   - Gauss-Legendre quadrature weights and evaluation points
quadrs2  - quadrature weights and evaluation points on subintervals

Ci       - cosine integral Ci(z)
Cin      - cosine integral Cin(z)
Si       - sine integral Si(z)
Gi       - Green's function integral

sinhc    - hyperbolic sinc function
asinhc   - inverse hyperbolic sinc function
sqrte    - evanescent SQRT for waves problems

flip     - flip a column, a row, or both
blockmat - manipulate block matrices
upulse   - generates trapezoidal, rectangular, triangular pulses, or a unit-step
ustep    - unit-step or rising unit-step function

dnv      - dn elliptic function at a vector of moduli
snv      - sn elliptic function at a vector of moduli
ellipK   - complete elliptic integral of first kind at a vector of moduli
ellipE   - complete elliptic integral of second kind at a vector of moduli
landenv  - Landen transformations of a vector of elliptic moduli
```

MATLAB Movies

grvmovie1 - pulse propagation with slow and negative group velocity ($v_g < 0$)
grvmovie2 - pulse propagation with slow and fast group velocity ($v_g > c$)
pulsemovie - step and pulse propagation on terminated transmission lines
pulse2movie - step propagation on two cascaded lines
RLCmovie - step getting reflected off a reactive termination
TDRmovie - fault location by time-domain reflectometry
xtalkmovie - crosstalk signals on coupled transmission lines
dipmovie - electric field pattern of radiating Hertzian dipole